2. Relations between two moving observers

It was already mentioned before in the introduction of this presentation, that in the following the Theory of Special Relativity (SRT) will be placed first in an axiomatic way to discuss general physical relationships. Using this basis, different combinations for the exchange of signals between two observers will be examined first. This will start with point-shaped observers before they will be looked at as containing an extended space. Subsequently the relations of angles between moving observers during the exchange of signals will be investigated.

The consequences derived will be discussed and compared with observations and calculations presented in the literature. It will be shown that the results do not contain any contradictions. Furthermore, additional considerations concerning the calculations of angles will be derived. These are based on geometric calculations and lead first to the expected result that a defined contraction of space must exist. It will also be shown that the contraction must be considered as symmetric in moving direction and opposite to it. This will become important later for the examination of alternative theories, which will be discussed in chapter 11.1.

Following the historical development, the participating observers performing experiments will first be specified as “at rest” and “moved”. In further considerations it will become clear, that these definitions in general can be replaced by “relatively moving against each other”. This approach is not used very often today, but sometimes it still can be found in new literature [21].

2.1 Exchange of signals between point-shaped observers

Although the first considerations and deductions presented here will be trivial at first sight, these simple approaches are already providing clear evidence of the limits of classical mechanics. To avoid discrepancies, it is even necessary for simple constellations, like these are valid for the exchange of signals between two point-shaped observers, to implement the calculations of the Lorentz-Transformation.

In the following this will be shown for some simple examples before more complex considerations will be discussed in detail.

2.1.1 Movement decreasing or increasing the distance

When two observers A and B decrease or increase their distance without acceleration, the transmission of light signals periodically emitted is of general interest. Following the classical theory according to Newton it is apparent, that the moved observer will detect a larger interval compared to the observer at rest, although the period of emission is the same for both (see Fig. 2.1).

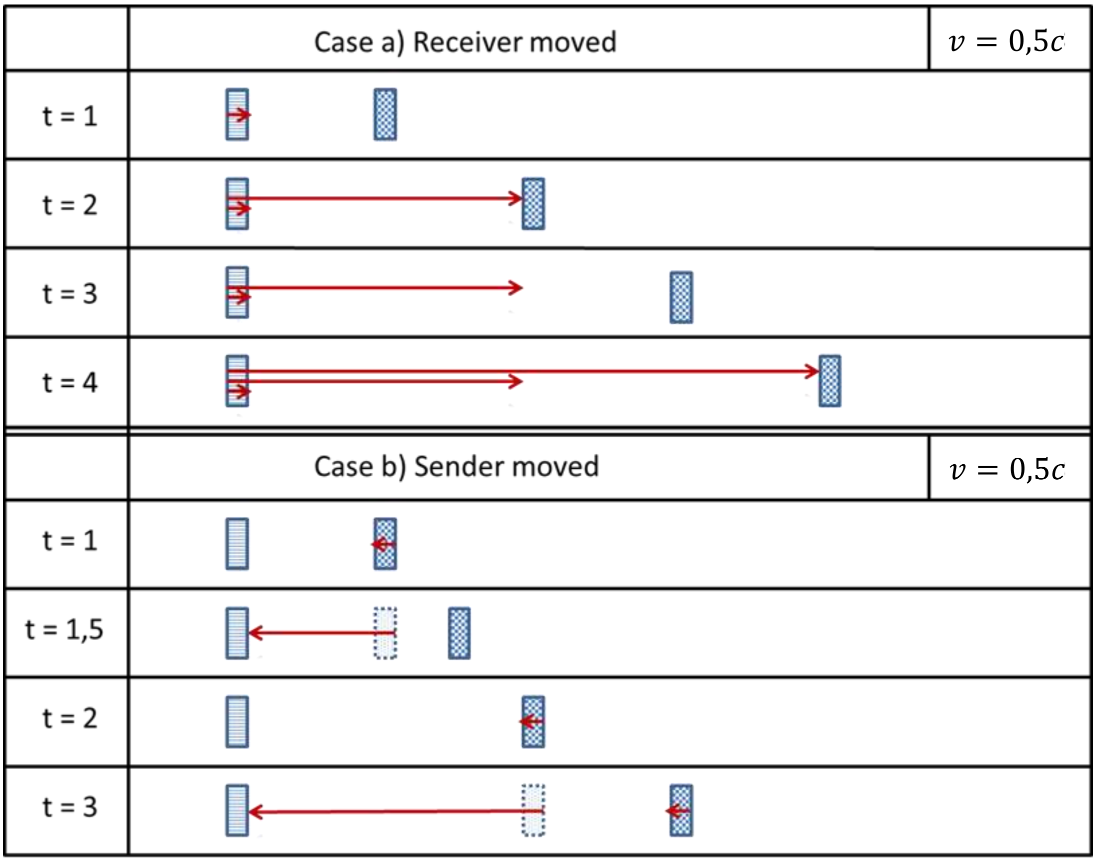


Fig. 2.1: Differences in the intervals of detected light signals by an observer at rest and a moving observer according to classical theory.
Observers have contact at $t = 0$,
Signal interval $\Delta t = 1\ TU$ (time unit),
Example for $v = 0{,}5c$

In this example with $v = 0{,}5\ c$ the moving observer would detect a signal every 2 time units (TU), whereas the observer at rest would find a difference of 1,5 TU. According to these considerations both observers would be able to calculate their velocity by the measurement of the signals from the partner. This is in clear contradiction to the experimental observation, that the results of trials like these are always independent of the state of motion.

In Fig. 2.2 the possibilities for the state of motion between a moved observer and an observer at rest are put together. Furthermore, in Tab 2.1 the fundamental relations are presented.

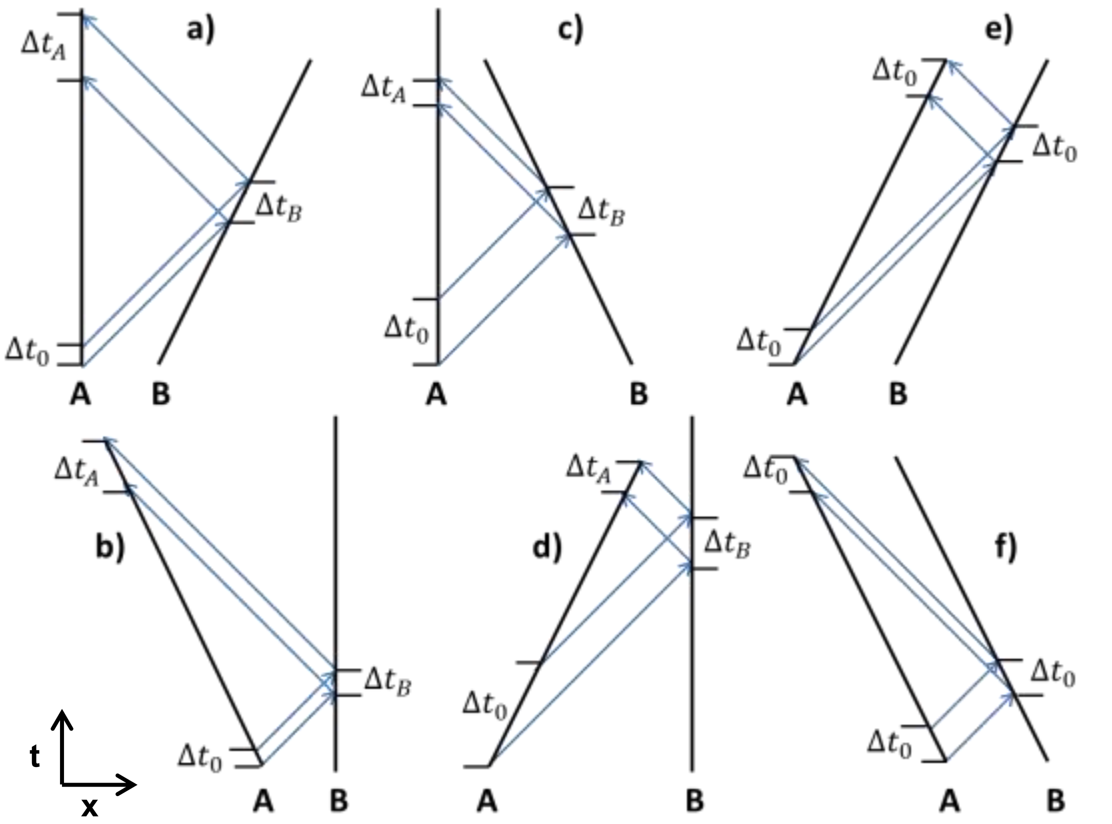


Fig. 2.2 Space-time diagrams for possibilities of light signal exchange

a)	$\Delta t_B = \Delta t_0 \frac{1}{1-\frac{v}{c}}$	c)	$\Delta t_B = \Delta t_0 \frac{1}{1+\frac{v}{c}}$	e)	$\Delta t_B = \Delta t_0$
	$\Delta t_A = \Delta t_0 \frac{1+\frac{v}{c}}{1-\frac{v}{c}}$		$\Delta t_A = \Delta t_0 \frac{1-\frac{v}{c}}{1+\frac{v}{c}}$		$\Delta t_A = \Delta t_0$
b)	$\Delta t_B = \Delta t_0 \left(1+\frac{v}{c}\right)$	d)	$\Delta t_B = \Delta t_0 \left(1-\frac{v}{c}\right)$	f)	$\Delta t_B = \Delta t_0$
	$\Delta t_A = \Delta t_0 \frac{1+\frac{v}{c}}{1-\frac{v}{c}}$		$\Delta t_A = \Delta t_0 \frac{1-\frac{v}{c}}{1+\frac{v}{c}}$		$\Delta t_A = \Delta t_0$

Tab. 2.1 Time intervals for the signal exchanges presented in Fig. 2.2

In the following the conditions for an exchange of light signals from A to B and vice versa according to Fig. 2.1 shall be presented in a simple space-time-diagram (see Fig 2.3). To realize this, the variations a) and b) from Fig. 2.2 will be combined.

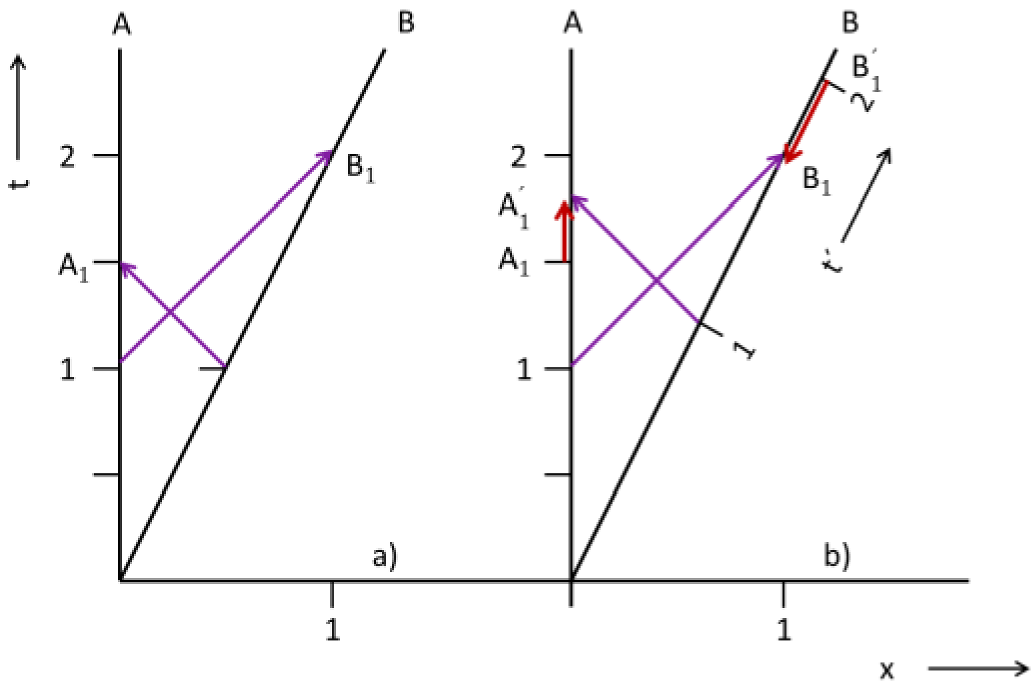


Fig. 2.3: Space-time-diagram for a signal exchange between observers A (at rest) and B (increasing the distance), Example for $v = 0{,}5c$
a) conventional (acc. to Galilei/Newton)
b) relativistic (acc. to Lorentz)

In case a) the conventional situation (acc. to Galilei/Newton) is presented. Both observers are emitting their signals at time $t = 1TU$ and these are detected at A_1 resp. B_1 by the partner. This diagram is valid e.g. for the exchange of acoustic waves, when A is at rest against a medium (i. e. air or water). But it was already mentioned before that this could not be detected by any experiments conducted using light signals.

Already at the end of the 19th century a solution for this (inside classical mechanics acc. to Newton existing) problem was presented by H. A. Lorentz. To realize this, it is necessary to assume, that at higher velocities an effect of time dilatation will be present. This means that time is running slower for the moved observer. This effect is integrated in part b) of the diagram. For observer B the time is running slower and therefore B is sending his signal later; this will arrive at the partner at A_1'. Because of the time dilatation the additional effect occurs that B is subjectively detecting the signal sent from A earlier. This effect is presented in the diagram by the transition from B_1 to B_1'.

The exact parameter of the time dilatation can be calculated in an easy way according to Fig. 2.2, cases a) and b). For the transition from a system at rest to a moved observer for Δt_0 the relation is valid

$$\Delta t_{AB} = \Delta t_0 \frac{1}{1 - \frac{v}{c}} \tag{2.01}$$

In opposite direction it is

$$\Delta t_{BA} = \Delta t_0 \left(1 + \frac{v}{c}\right) \tag{2.02}$$

To match Δt_{AB} and Δt_{BA} it is necessary to expand the equations (2.01) and (2.02) by the parameter γ (where Δt_{AB} will be smaller and Δt_{BA} will be larger) and the equations develop to

$$\frac{1}{\gamma} \cdot \frac{1}{1-\frac{v}{c}} = \left(1+\frac{v}{c}\right) \cdot \gamma \qquad (2.03)$$

with

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (2.04)$$

The parameter γ calculated here is the same as the Lorentz-Factor of Eq. (1.03).

It is therefore not possible for observers A and B to decide, whether they are moving or at rest. This implies that observer B also has the impression, that the time is running slow for A compared to his perception.

The example presented here for observers who increase their distance can also easily transformed to the view of observers which are approaching each other (see. Fig. 2.4, larger scale compared to Fig. 2.3).

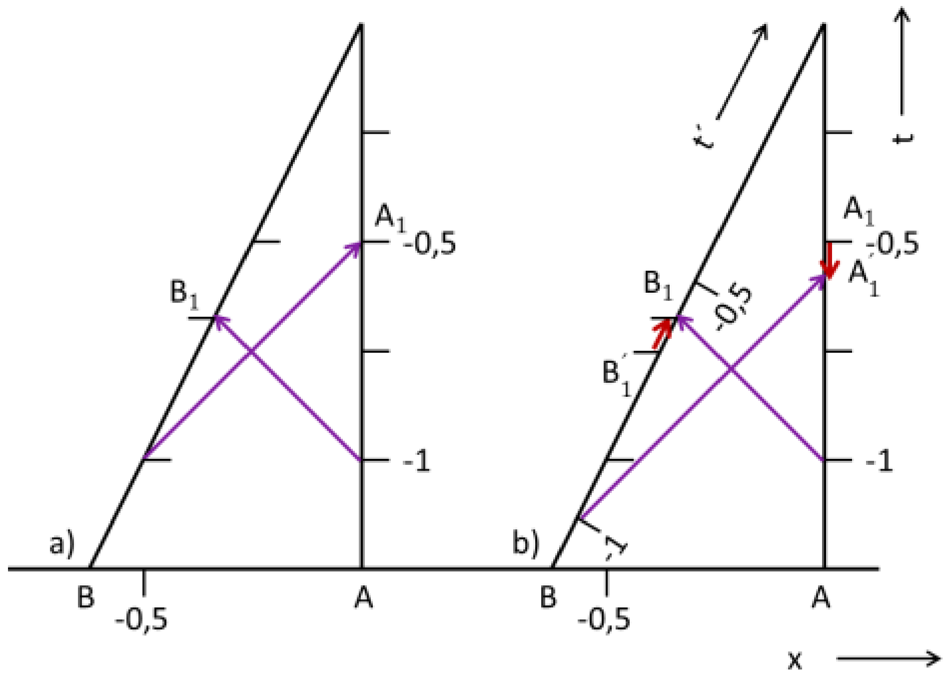


Fig. 2.4: Space-time-diagram for a signal exchange between observers A (at rest) and B (approaching), Example for $v = 0{,}5c$
a) Conventional (acc. to Newton)
b) Relativistic (acc. to Lorentz)

For the transition from a system at rest to a moving observer the time Δt_0 is according to case c) and d) presented in Fig. 2.2

$$\Delta t_{AB} = \Delta t_0 \frac{1}{1+\frac{v}{c}} \qquad (2.05)$$

and in opposite direction

$$\Delta t_{BA} = \Delta t_0 \left(1 - \frac{v}{c}\right) \tag{2.06}$$

The equations (2.05) and (2.06) must again be expanded by the parameter γ (Δt_{AB} smaller and Δt_{BA} larger) and it follows

$$\frac{1}{\gamma} \cdot \frac{1}{1+\frac{v}{c}} = \left(1 - \frac{v}{c}\right) \cdot \gamma \tag{2.07}$$

with the same result for γ as shown in Eq. (2.04).

It shall be stated again that the time dilatation of the moving observer is necessary to avoid discrepancies. Without this effect it would always be possible to distinguish a moving observer from an observer at rest by simple experiments.

2.1.2 Movement in arbitrary directions

It was established so far, that it is not possible for two observers increasing their distance or approaching each other to decide by measurements regarding the exchange of light signals whether they are moving or at rest. When the velocity vectors of the observers are not parallel, and they are passing by with the minimum distance a the situation changes, and more effort is necessary to verify that the observations of all participants are equivalent.

The following examination set-up shall be chosen:

1. Both observers will send out signals, the (subjective) interval is Δt.
2. For an incoming signal the angle referring to the direction of the sending observer is determined.
3. If the incoming signal is exact transverse to the moving direction of the sender a response signal with a special designation will be sent.
4. The signals are coded in a defined way to realize a final evaluation at the end of the trial. After the exchange of all data it is possible to find out, at what time the signals were sent which were detected as coming in exactly from the transverse direction.

First a moving observer B is considered, which is passing the observer at rest (A) in a minimum distance a with the speed v. In this case A will detect the signals sent from B in a (subjective) interval $\gamma\Delta t$. Compared to this observer B has a completely different view. Caused by the aberration effect B will measure the angle of the signal according to the equation

$$\delta = arcsin\left(\frac{v}{c}\right) = arctan\left(\gamma \cdot \frac{v}{c}\right) \tag{2 .10}$$

as coming in from the transverse direction (see Fig. 2.5). Here $v = 0{,}5c$ is chosen and the measured angle is $\delta = 30°$. Further discussions concerning the measurements of angles different to the transverse direction require additional geometric considerations which are presented in detail in chapter 2.3.

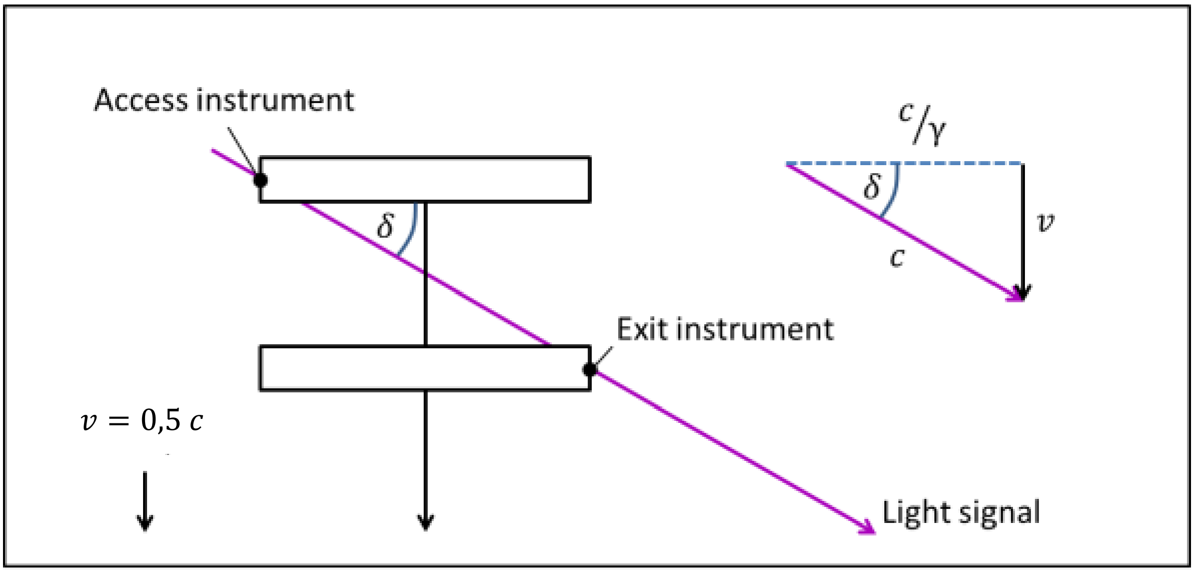


Fig. 2.5: Aberration effect: Measurement of angle δ caused by the movement of the receiver of a signal.

In the following it will be discussed, which values will be measured for the interval Δt and other relevant time measurements according to the situation presented in Fig. 2.6 for the moving observer and a system at rest.

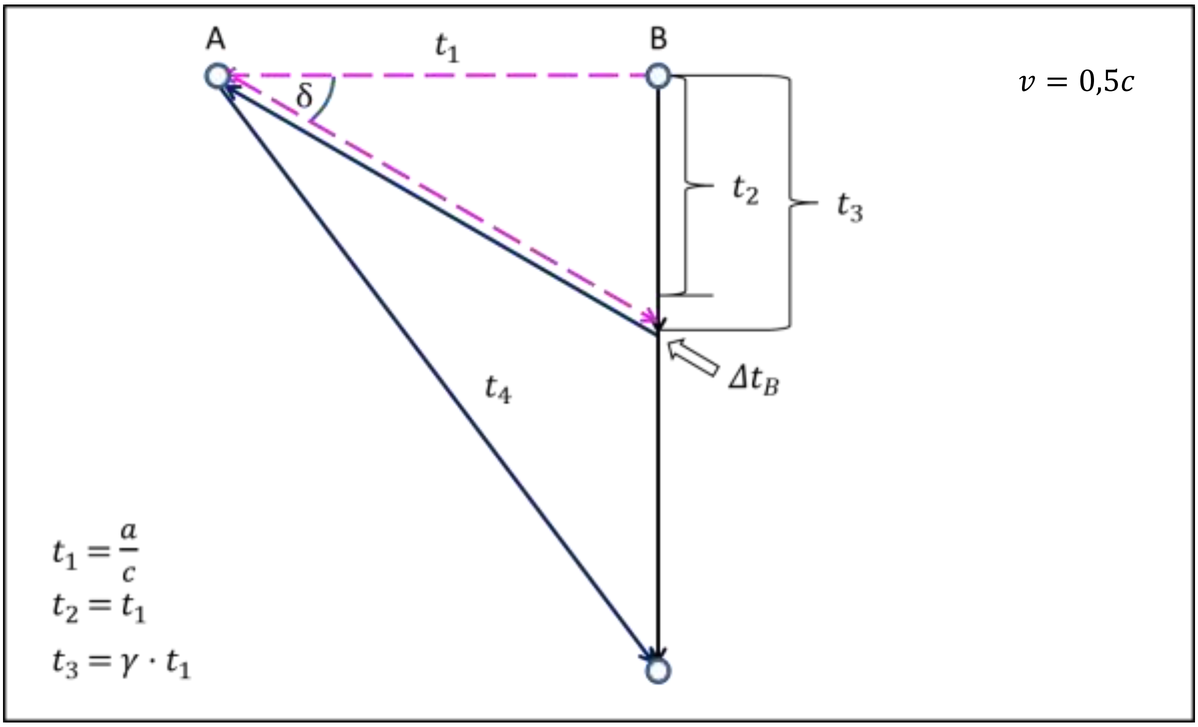


Fig. 2.6: Exchange of signals between A and B, example for $v = 0{,}5c$, $\delta = 30°$
Details for signal Δt_B: see Fig. 2.7; Total running time: Fig 2.8

a) Measurement of signal interval

As already shown the intervals between the signals emitted by the moving observer B will be measured by observer A at rest as $\Delta t_A = \gamma \Delta t$. This is caused by the effect of time dilatation valid for B.

The value Δt_B measured by B can be calculated using an approximation calculation according to the scheme presented in Fig. 2.7. At the beginning a signal is sent by A and this is received at point B_0, the next signal is following after time Δt_0. When it arrives at point B_0, then the observer has already moved on to point B_1 and the additional time for the extended way must be added. If it is presumed that $\Delta t_0 \ll t_1$ then it is possible for the calculation to shift the signals sent by A parallel in direction of B_1 without changing the value of δ. When the signal arrives at point B_1 then an additional movement to B_2 took place and the calculation must be repeated accordingly.

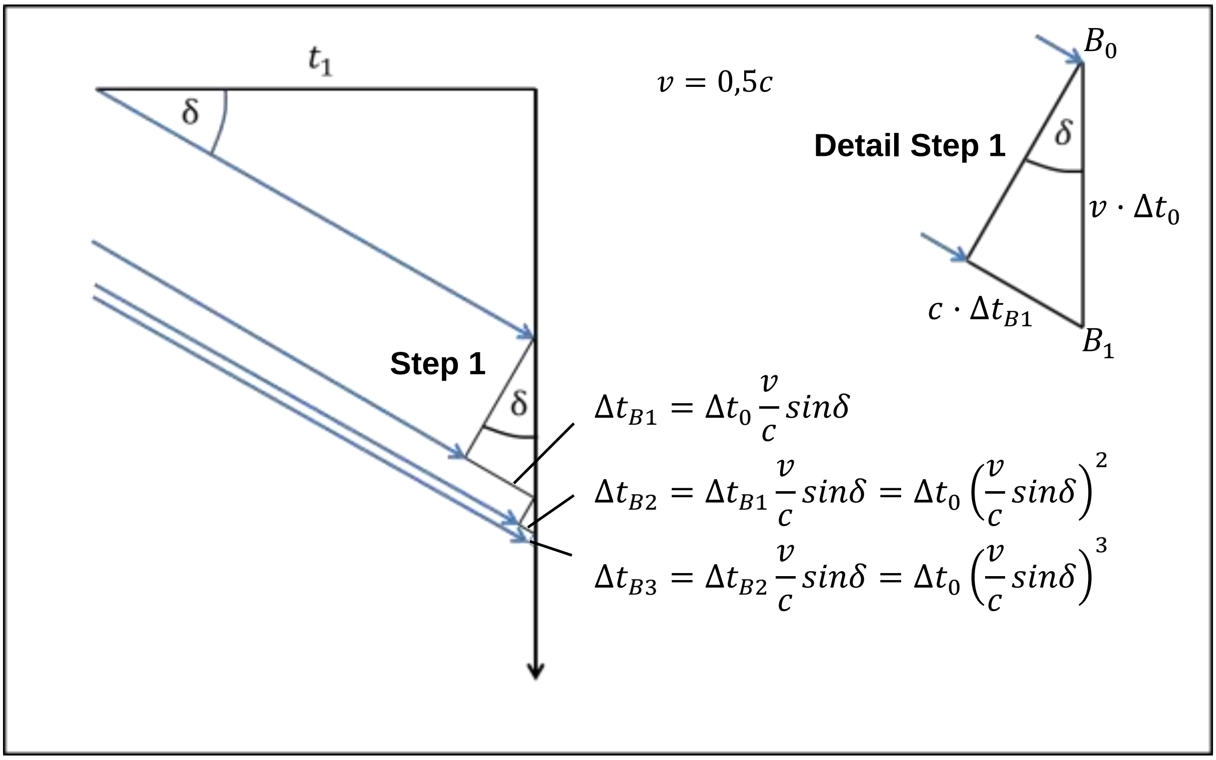


Fig. 2.7: Scheme for calculation of signal interval Δt_B (for $\Delta t_0 \ll t_1$). Presentation of the first 3 steps.

The single values can be summarized

$$\Delta t_B = \Delta t_0 + \sum_{i=1}^{\infty} \Delta t_{i-1} \frac{v}{c} sin\delta = \Delta t_0 \sum_{i=0}^{\infty} \left(\frac{v}{c} sin\delta\right)^i \tag{2.11}$$

In this case a geometrical series of the form

$$S_n = \sum_{i=0}^{n} q^i \tag{2.12}$$

is derived, where S_n is the limit value and

$$q = \frac{v}{c} sin\delta \tag{2.13}$$

With $n \to \infty$ and $q < 1$ it follows

$$S_\infty = \frac{1}{1-q} \tag{2.14}$$

Because B is subjectively realizing that the signal is arriving from the transverse direction Eq. (2.10) is valid

$$sin\delta = \frac{v}{c} \tag{2.15}$$

Hence

$$S_\infty = \frac{1}{1-\frac{v^2}{c^2}} = \gamma^2 \tag{2.16}$$

The combination with (2.11) reveals

$$\Delta t_B = \gamma^2 \cdot \Delta t_0 \tag{2.17}$$

The calculation shows that the moving observer B will measure (subjective) a value of $\gamma\Delta t$, because he is subject to time dilatation himself. Thus, it is verified that observers A and B are measuring the same values for the intervals of incoming signals.

b) Measuring of total running time of signals

The running time of a signal emitted by A and identified by B as transverse to his moving direction is γt_1 (see Fig. 2.6).

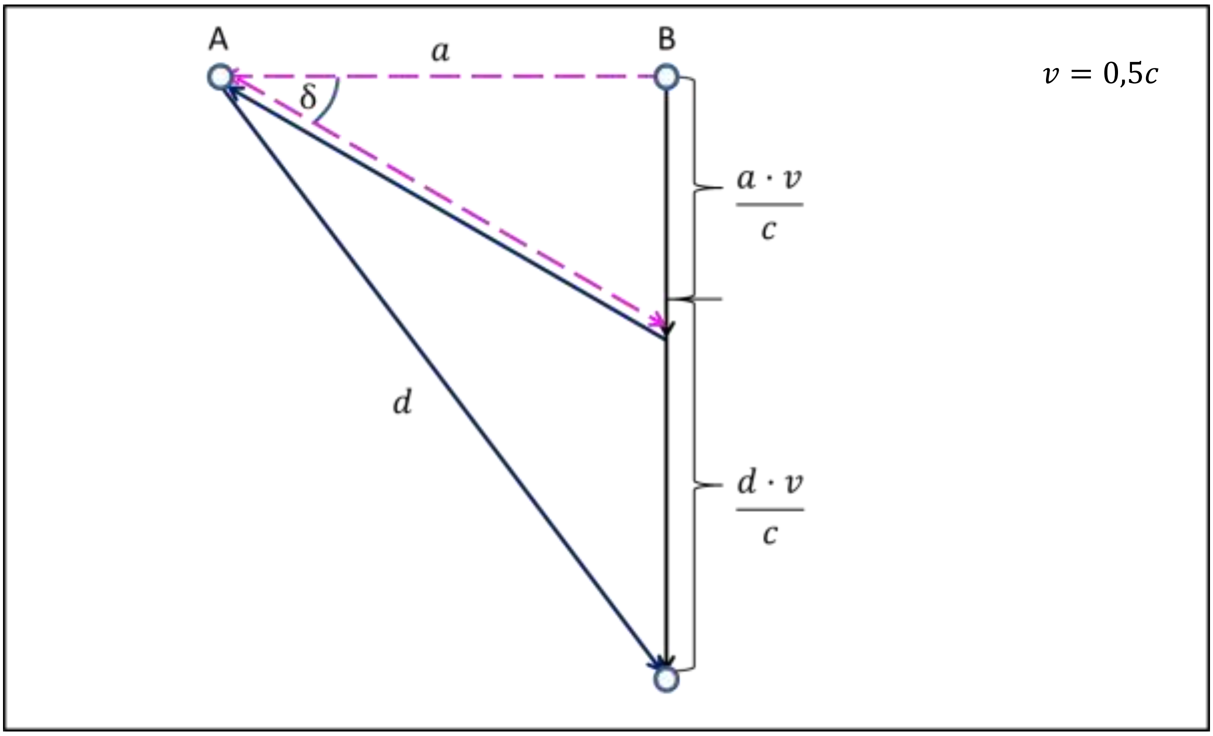


Fig. 2.8: Signal path B → A → B and definition of distances travelled.

Because B is sending the signal back the same way the total running time is $2\gamma t_1$. For B the first value is t_1 (see Fig. 2.6), the way back t_4 must be calculated. To do this some important definitions are necessary (see Fig. 2.8).

The distance d (corresponding to the time t_4) is derived by

$$a^2 + \left(\frac{v}{c}a + \frac{v}{c}d\right)^2 = d^2 \tag{2.18}$$

Completing the square shows

$$a = d\left(-\frac{v^2}{c^2\left(1+\frac{v^2}{c^2}\right)} \pm \sqrt{\left(\frac{v^2}{c^2\left(1+\frac{v^2}{c^2}\right)}\right)^2 + \frac{1-\frac{v^2}{c^2}}{1+\frac{v^2}{c^2}}}\right) \quad (2.19)$$

Considering only positive values, it is achieved after simplification

$$a = d\left(-\frac{v^2}{c^2\left(1+\frac{v^2}{c^2}\right)} + \sqrt{\frac{v^4}{c^4} + \left(1+\frac{v^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}\right) \quad (2.20)$$

$$= d\left(-\frac{v^2}{c^2\left(1+\frac{v^2}{c^2}\right)} + 1\right) = d\left(\frac{1-\frac{v^2}{c^2}}{1+\frac{v^2}{c^2}}\right) \quad (2.21)$$

and

$$d = a\left(\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right) \quad (2.22)$$

For calculation of the total distance the value of a is added

$$d + a = a\left(\frac{1+\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}} + 1\right) = a\left(\frac{1+\frac{v^2}{c^2}+1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right) = 2a\gamma^2 \quad (2.23)$$

The calculations lead to a total time of $2\gamma^2 t_1$ and therefore to a difference of factor γ between observers A and B which is compensating the time dilatation for the moving observer B. It is shown again that identical subjective measurements are valid.

2.2 Exchange of signals inside moving bodies

The considerations taken so far illustrate the fundamental relations during experiments concerning an exchange of signals between observers at different speed. Doing this, the conditions are, however, not fully described without discrepancies. If for example an observer at rest could directly monitor measurements of the speed of light between two moving observers, he would find differences between his results compared to the results of the other observers without further modification. This would cause a violation of the fact, that measurement of the speed of light show the same results in any inertial system. It has to be mentioned that here differences for the results in moving direction and in other arbitrary directions occur; in the following these cases will be treated separately.

In the following only the exchange of light pulses will be part of the calculations. The discussion of light as a wave and the special characteristics connected with this feature require special considerations and will be presented in chapter 8.

2.2.1 Exchange of signals in moving direction

For the presentation of this situation the time for the exchange of signals between observers A and B shall be investigated.

While the time in a system at rest for going and coming is

$$t_{AB} + t_{BA} = 2t_0 \tag{2.30}$$

it is different for moving objects for observations from a system at rest (see Eq. 2.01 and 2.05)

$$t_{AB} + t_{BA} = t_0 \frac{1}{1 - \frac{v}{c}} + t_0 \frac{1}{1 + \frac{v}{c}} \tag{2.31}$$

with

$$t_{AB} + t_{BA} = t_0 \left[\frac{\left(1 + \frac{v}{c}\right) + \left(1 - \frac{v}{c}\right)}{\left(1 + \frac{v}{c}\right)\left(1 - \frac{v}{c}\right)} \right] = 2\gamma^2 t_0 \tag{2.32}$$

It was already mentioned before that the time for moved observers is enlarged by the parameter γ. During the above-mentioned calculation, the spatial extension is reaching, however, the factor γ^2. To overcome this contradiction, it is necessary to reduce in addition the distance between the two observers by the factor γ. This reduction is generally named “space contraction”.

When the effects of time dilatation and space contraction are considered together all discrepancies disappear. It is worth mentioning, that the times for travelling the distances between A → B and B → A are different in view of a system at rest, but that the summation of the times (when time dilatation is considered) is leading to the same result compared to a system at rest.

These correlations are not only valid for the observer at rest. The moved observer also will find during the evaluation of own measurements concerning the distances in the system at rest that these are contracted by the factor γ. Time dilatation and space contraction are thus depending on each other to create a physical frame without discrepancies.

A simple example shall demonstrate the results. A case shall be monitored where observers A and B are placed in a system with a constant distance a. At time 0 observer A is sending out a signal to B which is immediately reflected to A. When A and B are viewed as at rest, the distances of going and coming and the connected times for the transport of the signal are equal in both directions. If both observers are moving constantly in relation to a different inertial system, however, the situation is completely different. This shall be demonstrated in a space-time-diagram (Fig. 2.9). For simplification of the presentation the values are normalized. This means that $a = 1$, in addition the time t is converted to ct and is – as valid for the space values x – standardized to a value of 1. (The use of ct instead of t is frequently used; in this case the dimensions of x and ct are identical and it is easily possible to take direct readings out of the diagram).

Calculations analog Eq. (2.32) lead to

$$x_T = x_1 - x_2 = \frac{a}{\gamma\left(1-\frac{v}{c}\right)} + \frac{a}{\gamma\left(1+\frac{v}{c}\right)} = \frac{2\gamma v a}{c} \tag{2.33}$$

$$t_T = t_1 + t_2 = \frac{a}{c\gamma\left(1-\frac{v}{c}\right)} + \frac{a}{c\gamma\left(1+\frac{v}{c}\right)} = \frac{2\gamma a}{c} \tag{2.34}$$

Inserting these values into the Lorentz-Equations Eq. (1.07) and (1.08) the results $x' = 0$ and $t' = 2\,a/c$ will appear which are the expected findings for observers at rest. At this stage of the discussion it is not clear, how the Lorentz-Equations can be derived; in chapter 3.3 different methods will be presented in which way this is possible.

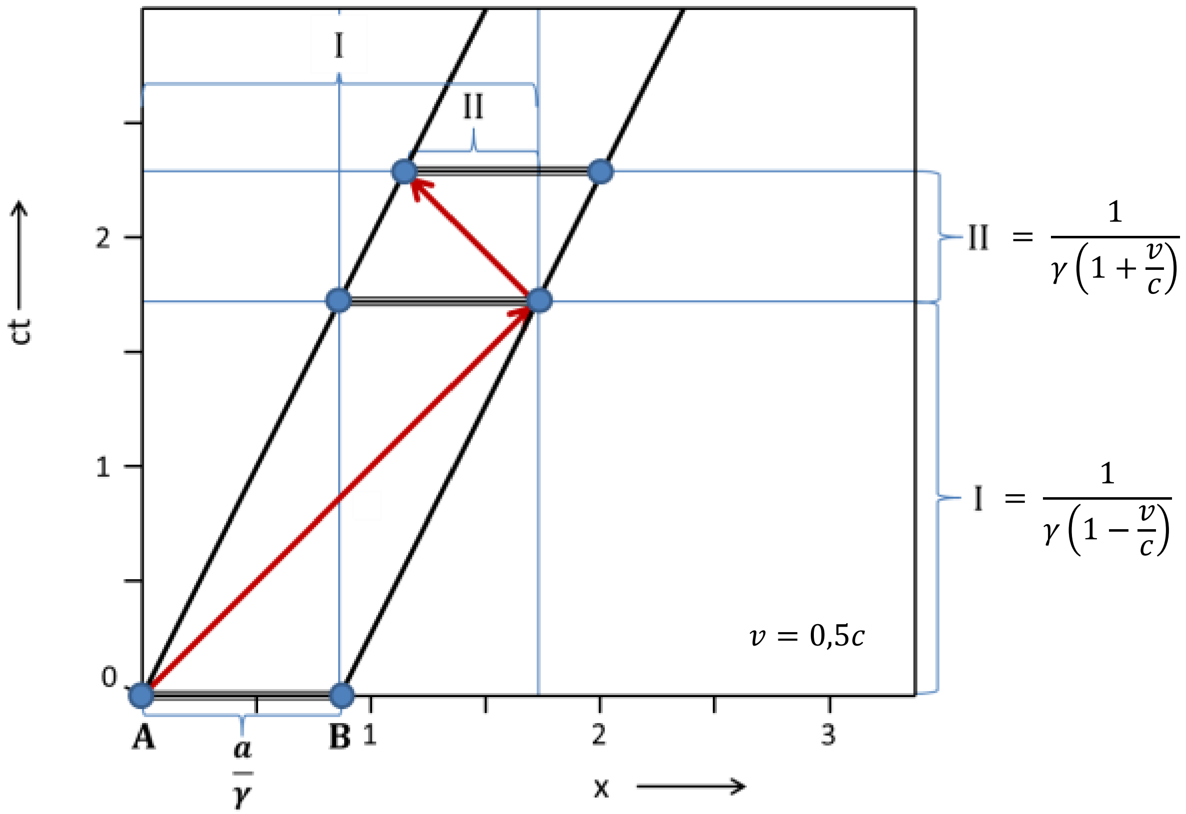


Fig. 2.9: Exchange of signals between observers A and B (marked using red arrows) in a moving system. Example for $v = 0{,}5c$

2.2.2 Exchange of signals during passing of two observers

When a more complex approach for the observations is considered, like it is the case for measurements between identical laboratories, which are passing in a close distance and exchanging light signals between front and back end, also no deviations will occur. An example shall be discussed in detail.

The experimental set-up is the following:

1. Two identical laboratories with observers A, B, C and A′, B′, C′ shall be prepared. The orientation is presented in Fig. 2.10. The positions of C and C′ are situated exactly in the middle of the laboratories.

2. The laboratory with A′, B′, C′ is moved relative to A, B, C according to the presentation in the diagram.
3. The moved laboratory is passing the observers at rest in a minimum distance to keep aberration effects as small as possible.
4. As soon as the observers of both systems pass each other signals to C resp. C′ will be sent. C resp. C′ are reflecting the signals to the sender and are recording the relevant periods.

At first observers A′ and A are passing. For small velocities (compared with the speed of light) the passing of B′ and A plus also A′ and B will happen simultaneously. When relativistic velocities are used, however, this will not be the case. Here the moved system will show a contraction in moving direction and the contacts between the observers will happen at different times. At the end B′ and B will pass. In total there are 4 different situations for contacts, which are presented in Fig. 2.11 in a space-time-diagram.

After the end of the experiment the corresponding time records between all observers shall be compared. For the selected example with the velocity $v = 0{,}5\ c$ the coordinates for C and C′ are presented in table 2.2. In addition to the values from the experiment the calculated results determined by the Lorentz-Transformation are also presented in this table. The space and time coordinates will be discussed in the following to allow an exact comparison between the different situations.

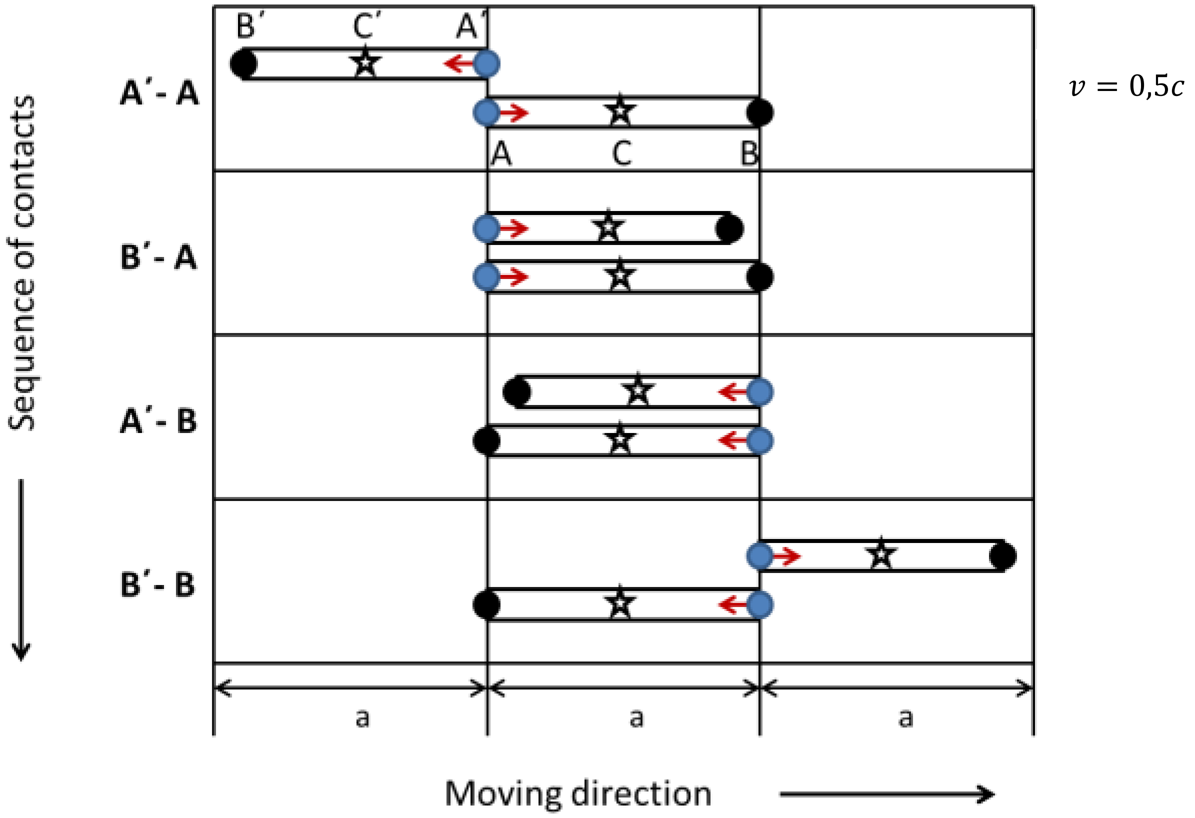


Fig.:2.10: Laboratory with observers A and B to transmit signals and C to receive. An identical laboratory with observers A′, B′ and C′ is passing with the velocity $v = 0{,}5\ c$. During all contacts of A and B with A′ and B′ a signal is transmitted and received by C and C′.

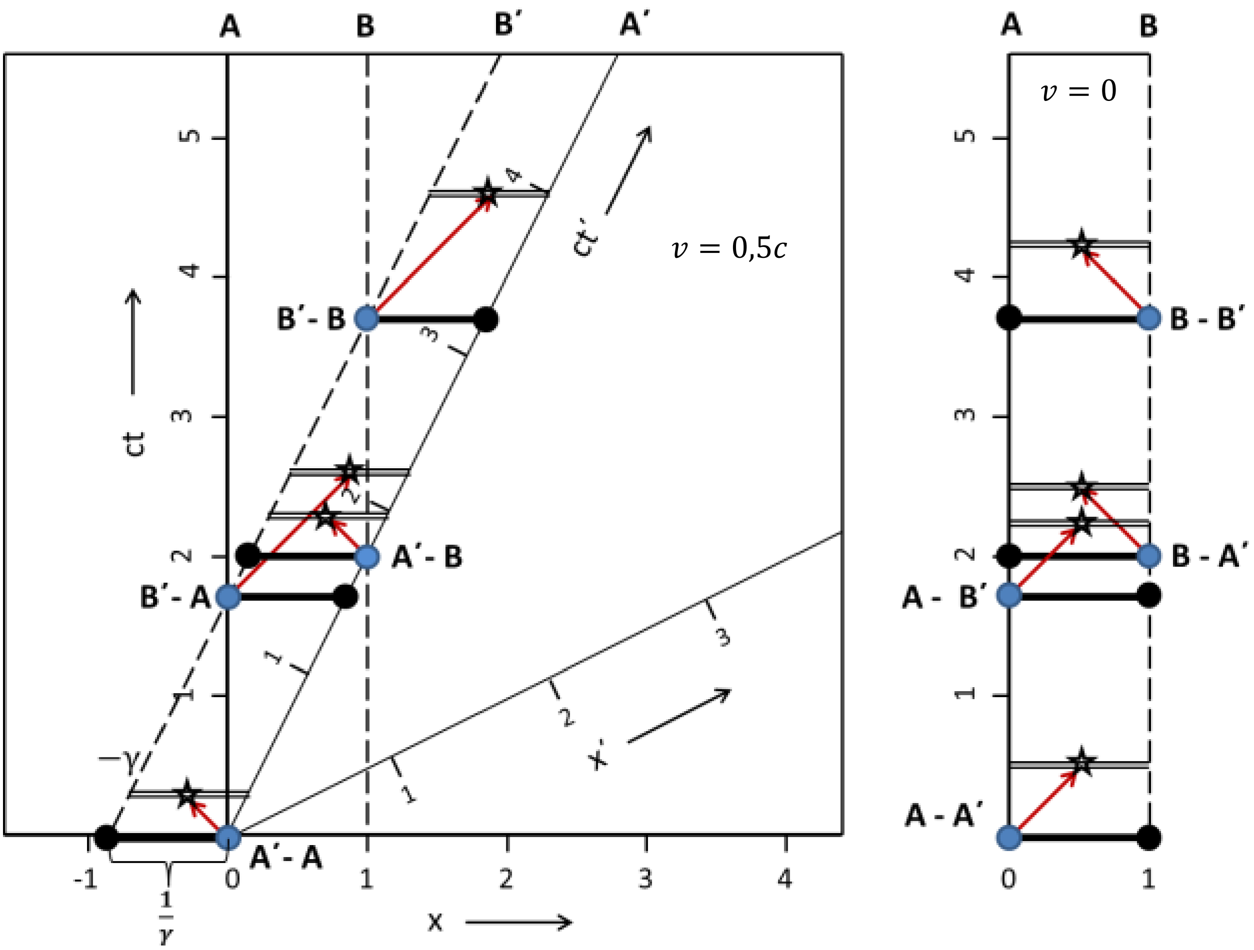


Fig. 2.11: Time sequence of received signals in the middle of two identical laboratories; signals are transmitted when passing.
Left: Moving laboratory
Right: Laboratory at rest

Case	Observer	A/A′	A/B′	B/A′	B/B′
1	C	[0,5 ; 0,5]	[0,5 ; 2,232]	[0,5 ; 2,5]	[0,5 ; 4,232]
2	C	[-0,289 ; 0,289]	[0,866 ; 2,598]	[0,711 ; 2,289]	[1,866 ; 4,598]
3	C′	[-0,5 ; 0,5]	[-0,5 ; 2,5]	[-0,5 ; 2,232]	[-0,5 ; 4,232]

Tab. 2.2: Coordinates for space [bracket left] and time [bracket right] for the experiment according to Fig. 2.11.
Line 1: Values for the observer at rest
Line 2: Observation by the observer at rest regarding the moving system
Line 3: Calculated values for the moved observer according to the Lorentz-Transformation

Coordinates of space

It is clear at first sight that the coordinates of space in the first line must be constant. The chosen parameters lead to a value of 0.5.

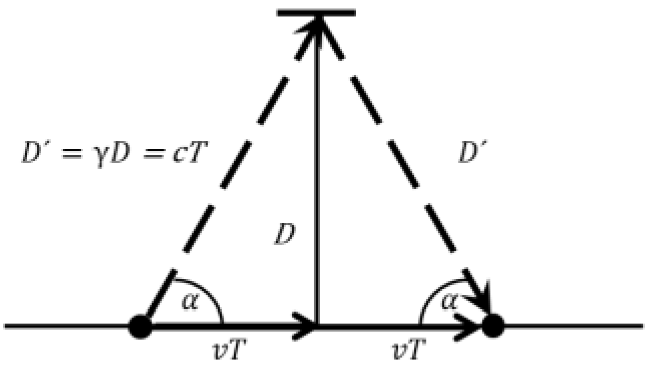
For the moved system, the parameters vary depending on the geometrical relations according to line 2. The values of the coordinates of space derived by calculations using the Lorentz-Transformations are equal to those of the system at rest with the only difference that the algebraic sign is negative. This means, that the observers at rest and in the moved system are measuring the same values.

Coordinates of time

The coordinates of time show a similar effect. In this case the situation is different, however, because for C and C′ the values of A/B′ and B′/A are exchanged. It is obvious, that the principle of relativity requires, that C resp. C′ must receive the signal of "their" observer A resp. A′ first. It is important, that for the observer at rest the change in the values of time is necessary to show a proper sequence of contacts between A′ and B′ to C′. So, this short summary provides clear evidence that no differences between measurements of all observers taking part will appear.

2.2.3 Exchange of signals in arbitrary directions

In the following the situation shall be discussed, that a signal is transmitted and reflected transverse to the moving direction (i. e. y-direction). The time dilatation occurring for the moving observer, which travels the distance of $d = vT$ when the signal reaches the reflector, is exactly compensated by the longer path of the signal $D' = cT$ (Fig. 2.12). This means that it is not possible for the moving observer to find a difference compared to the situation at rest and so again no violation of the principle of relativity can be found.



$$tan\alpha = \frac{D}{vT} = \frac{c}{\gamma v}$$

Fig. 2.12: Signal exchange transverse to the moving direction

In contradiction to the effects of a longitudinal signal exchange this means, that in the view of an observer at rest in transverse direction there is a change in the transmission angle because of aberration. The value can be calculated as presented in Fig. 2.12 using the tangent value (see also Eq. (2.11) with $\alpha = 90° - \delta$).

Whereas the situation concerning the exchange of signals in direction of a moving observer was discussed first, the behavior in transverse direction is described here. No discrepancies to the expected circumstances for the observer in motion appear and the principle of relativity is respected in any case.

To start with the next step discussing the observations during signal exchange in any arbitrary spatial direction it is necessary first, to start with basic considerations concerning the dependencies between the angles of incoming and outgoing signal due to aberration for moved observers in view of a reference system at rest. This will be presented in the following; afterwards, using these derivations, it will be shown that no differences appear between the subjective measurements in a system at rest and for a moved observer. This issue will be discussed in chapter 2.4 and the validity will be proven by calculations of an example using a spere where light signals start from the center and return after reflection.

2.3 Exchange of signals and correlation of angles

In the following it shall be investigated, which correlations appear when emitted and received signals have different directions compared to a moving body. This effect is commonly referred to as aberration (see Fig. 2.5).

As already discussed in detail, the relativistic approach to calculations of a moved observer requires the consideration of the effect, that the body will be contracted in moving direction. Up to now this effect was only treated as a summation of going and coming of the signal and first nothing is known about the splitting into the single trips. Out of the principle of relativity it can be deduced, however, that this contraction must be symmetric to the middle axis of the moved body according to Fig. 2.13. It makes no difference in which direction the movement will take place.

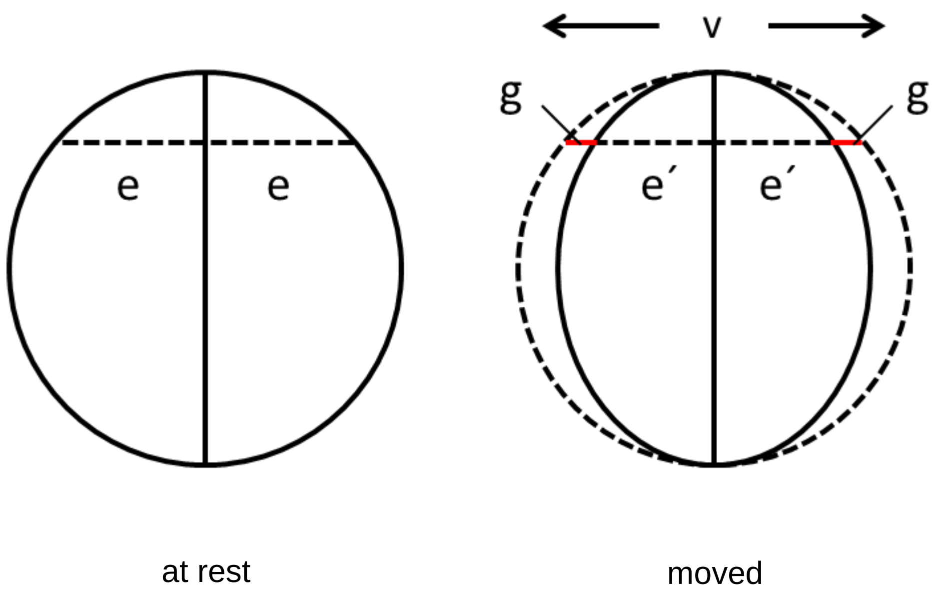


Fig. 2.13 Contraction of a moved body

In this case the distance e' in the moved system is equal to $e - g$ or e/γ.

2.3.1 Reception in a moving body

In the following the values for the reception in a moving body will be investigated. First it is necessary to define the exact conditions for the analysis. The following set-up shall be used:

A sphere with the radius a contains holes in the circumference in adequate quantity where adjusted light beams can enter (i. e. at point P_1, see Fig. 2.14). When such a beam is touching the center (P_2), then the observer can define the corresponding angle using geometric evaluations. Any of these holes relates to an angle of α' resp. β' because of the geometrical definitions of the exact position and the radius a.

If the observer receiving the signal is moving, then an observer at rest will find different angles for the incoming signal and his measurements will be α resp. β. In his view the signal will travel a distance d inside the system. For the calculations it has to be considered that, as already stated before, the sphere will be deformed in moving direction (see Fig. 2.13). In this case for the incoming signals the geometric dependencies are defined according to Fig. 2.14. The incoming direction from behind (part a) leads to the following dependencies

$$d^2 = f^2 + (e + b - g)^2 \tag{2.40}$$

and

$$f = d \cdot sin\alpha \qquad f = a \cdot sin\alpha' \tag{2.41}$$

Further

$$e = a \cdot cos\alpha' \tag{2.42}$$

$$\frac{b}{v} = \frac{d}{c} \tag{2.43}$$

$$e - g = \frac{e}{\gamma} \tag{2.44}$$

The first calculation yields

$$a = d \cdot \frac{sin\alpha}{sin\alpha'} \tag{2.45}$$

Eq. (2.40) is developing to

$$d^2 = (d \cdot sin\alpha)^2 + \left(d\frac{v}{c} + d\frac{cos\alpha' \cdot sin\alpha}{\gamma \cdot sin\alpha'}\right)^2 \tag{2.46}$$

$$1 - sin^2\alpha = cos^2\alpha = \left(\frac{v}{c} + \frac{sin\alpha}{\gamma \cdot tan\alpha'}\right)^2 \tag{2.47}$$

$$tan\alpha' = \frac{sin\alpha}{\gamma\left(\pm\, cos\alpha - \frac{v}{c}\right)} \tag{2.48}$$

where because of geometrical considerations only positive values for $cos\alpha$ are valid. If the signal is approaching from the front (Fig. 2.14b) the relations are

$$d^2 = f^2 + (e - b - g)^2 \tag{2.49}$$

After the same calculation as presented before this leads to

$$tan\beta' = \frac{sin\beta}{\gamma\left(cos\beta + \frac{v}{c}\right)} \tag{2.50}$$

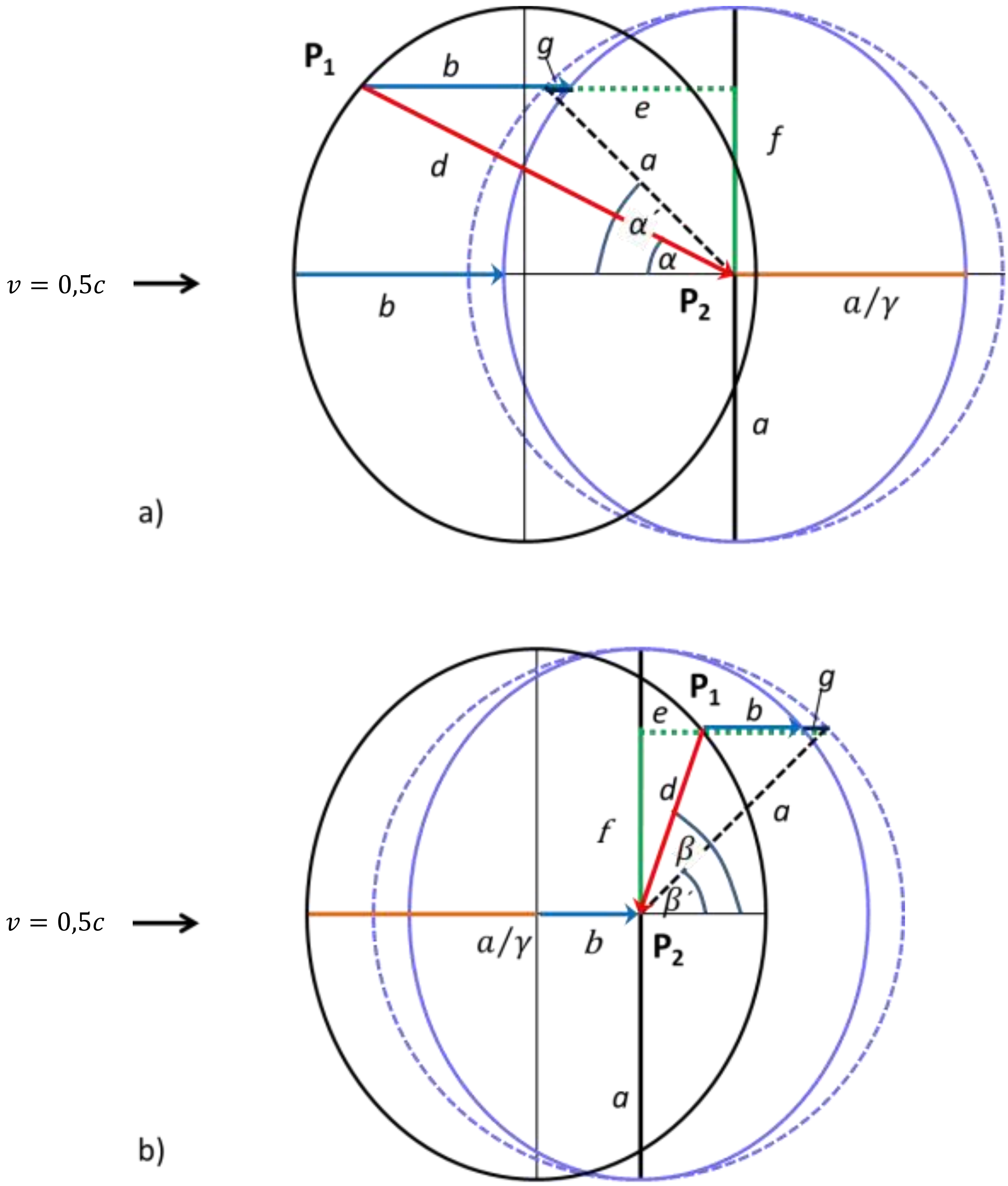


Fig. 2.14: Definition of parameters to determine the angle of incoming beams for a moved observer (examples for $v = 0{,}5c$ and $\alpha', \beta' = 45°$)
a) Signal approaching from behind, b) Signal approaching from the front

Before reviewing the results, the opposite situation with an outgoing light beam shall be discussed first.

2.3.2 Outgoing signals of moving bodies

For outgoing signals similar correlations apply. The relevant parameters are presented in Fig. 2.15. In this case the signal will be emitted from the center (P_1) and is passing a hole in the circumference of the sphere (P_2). In this case the space contraction of the moving body has also to be considered.

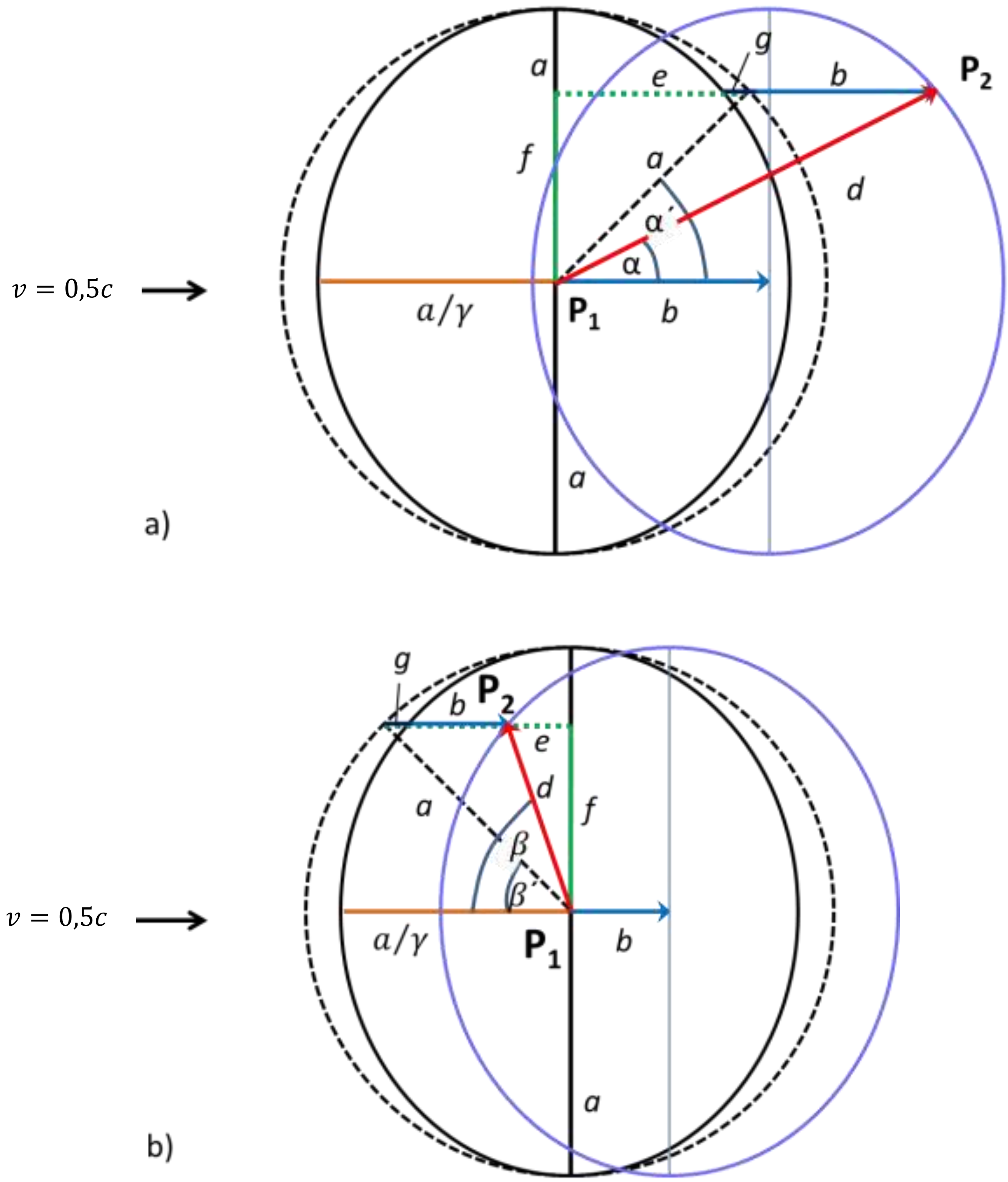


Fig. 2.15: Definition of parameters to determine the angle of outgoing beams for a moving observer (examples for $v = 0,5c$ and $\alpha', \beta' = 45°$)
a) Signal emitted in moving direction, b) Signal emitted backwards

For outgoing signals in moving direction (Fig. 2.15a) the results are exactly the same compared to incoming signals approaching from behind, which are covered by the equations presented from Eq. (2.40) to Eq. (2.48). For outgoing signals emitted backwards (Fig. 2.15b) the opposite combination occurs, and the result is Eq. (2.50) corresponding to the signal approaching from the front end.

2.3.3 Results of calculations of angles

At first it shall be demonstrated for the example discussed in chapter 2.1.2, that the results for a moved observer and a system at rest are exactly the same. To realize this, the propagation of the signals and the connected angles will be investigated. In view of the observer at rest (marked as "A") the process will start sending the signal 1 to observer B, following this, the signal 2 will be detected and returned, at the end the reflection of signal 1 is arriving. The angles of outgoing signals are marked with ε, whereas incoming signals carry the letter δ.

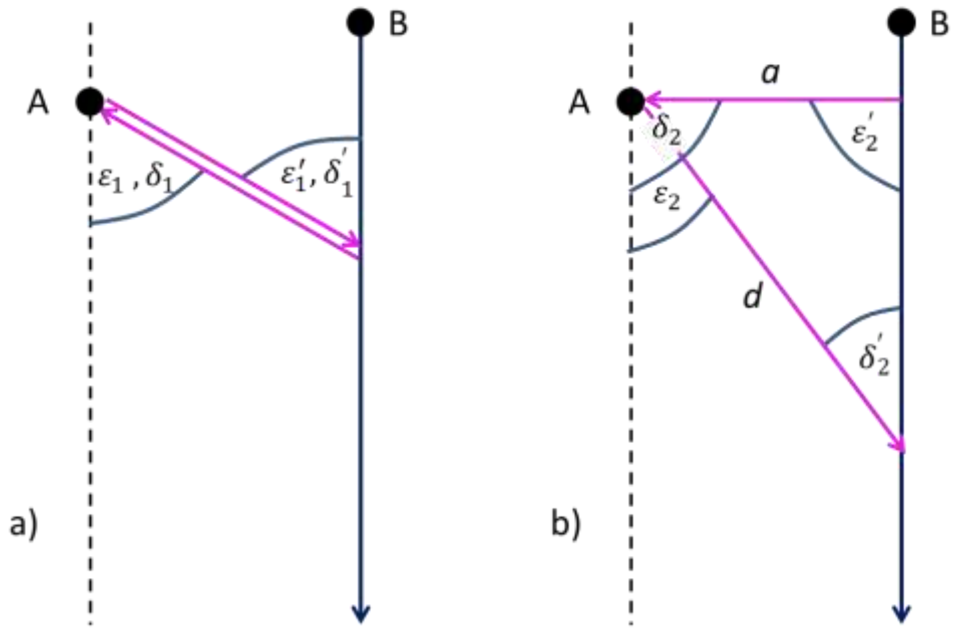


Fig. 2.16: Signal propagation according to situation in chapter. 2.1 with corresponding angles, example for $v = 0{,}5c$

Due to the chosen conditions the following situation is defined:

- The angles for incoming signals δ_2 and δ_1' are 90°.
- The values for incoming signal δ_1 and outgoing signal ε_1 are equal.
- The outgoing signal ε_2 can be calculated using Eq. (2.23) as

$$\varepsilon_2 = arcsin\left(\frac{a}{d}\right) = arcsin\left(\frac{1-\frac{v^2}{c^2}}{1+\frac{v^2}{c^2}}\right) = 36{,}87° \qquad (2.51)$$

Calculations for the chosen speed of $v = 0{,}5c$ show the following results:

	Initial value	Calculation	Result
1	$\delta_1' = 90°$	$tan\delta_1' = \frac{sin\varepsilon_1}{\gamma\left(cos\varepsilon_1 + \frac{v}{c}\right)}$	$\varepsilon_1 = 60°$
2	$\delta_2 = 90°$	$tan\varepsilon_2' = \frac{sin\delta_2}{\gamma\left(cos\delta_2 + \frac{v}{c}\right)}$	$\varepsilon_2' = 60°$
3	$\varepsilon_2 = 36.87°$	$tan\delta_2' = \frac{sin\varepsilon_2}{\gamma\left(cos\varepsilon_2 - \frac{v}{c}\right)}$	$\delta_2' = 60°$
4	$\delta_1 = 60°$	$tan\varepsilon_1' = \frac{sin\delta_1}{\gamma\left(cos\delta_1 + \frac{v}{c}\right)}$	$\varepsilon_1' = 36.87$

Tab. 2.3: Calculation of angles for the situation corresponding to Fig. 2.16

It is shown here that A and B find the same values for outgoing (60°; 36.87°) and incoming signals (90°; 60°). It is thus demonstrated that the principle of relativity is also valid for measurements of angles and that the spatial contraction must be symmetric to the middle axis of the moved body in moving direction and vice versa.

2.3.4 Literature review and evaluation

The following simple derivation of the aberration formula for relativistic velocities was presented by D. Giulini [19]. Here the emission of a light pulse from an observer with the coordinates x_0 and y_0 in a system at rest resp. x_0' and y_0' for a system moving with the velocity v is investigated in relation to their relative point of origin. In this case δ and δ' are the angles to the x-axis. At the time $t = t_0 = t_0'$ the systems meet in their respective points of origin. In this case the component u_x in the system at rest can be calculated using

$$u_x = -c \cdot cos\delta \tag{2.60}$$

and in the moving system

$$u_x' = -c \cdot cos\delta' \tag{2.61}$$

Integrated in the equation of relativistic addition of velocities

$$u_x' = \frac{u_x + v}{1 + \frac{u_x \cdot v}{c^2}} \tag{2.62}$$

the calculation yields

$$cos\delta' = \frac{cos\delta - \frac{v}{c}}{1 - \frac{v}{c} \cdot cos\delta} \tag{2.63}$$

Further comprehensive derivations of the calculations are leading to the same results (e. g. presented by R. K. Pathria [27]). Other investigations, however, show additional derivations, e. g. [28,89a]

$$sin\delta' = \frac{sin\delta}{\gamma\left(1-\frac{v}{c}\cdot cos\delta\right)} = \frac{\left(1-\frac{v^2}{c^2}\right)^{1/2} sin\delta}{1-\frac{v}{c}\cdot cos\delta} \tag{2.64}$$

A particularly useful formula is derived using the general valid formula for the tangent [19,28] yielding

$$tan\left(\frac{\theta}{2}\right) = \frac{sin\theta}{1+cos\theta} \tag{2.65}$$

Inserting equations Eq. (2.63) and Eq. (2.64) the transformation leads to

$$tan\left(\frac{\delta'}{2}\right) = \frac{sin\delta}{\gamma\left(1+\frac{v}{c}\right)(1+cos\delta)} \tag{2.66}$$

$$tan\left(\frac{\delta'}{2}\right) = \left(\frac{c-v}{c+v}\right)^{1/2} tan\left(\frac{\delta}{2}\right) \tag{2.67}$$

Using this equation, it is possible to determine in an easy way the value of δ depending on δ'. In the following, some selected results for all equations are calculated and compared. It must be considered that inverse functions (arc) for values between 0 and 180° are not exactly defined in cases where a sinus is present. The reason is, that in contrast to the co-sine, which is monotonously decreasing in this interval, the sine wave shows a maximum at 90° and therefore the inverse function contains two possible solutions. This is the reason why for angles $> 90°$ the standard result must be converted as presented in tables 2.4 and 2.5. (The tangent is monotonously increasing between 0 and 90°, which is sufficient acc. to Eq. (2.67), because when taking $\delta/2$ as argument the necessary interval is halved).

1: $\alpha' = arctan\left(\frac{sin\alpha}{\gamma\left(cos\alpha-\frac{v}{c}\right)}\right)$	**2**: $\alpha' = arccos\left(\frac{cos\alpha-\frac{v}{c}}{1-\frac{v}{c}\cdot cos\alpha}\right)$
3: $\alpha' = arcsin\left(\frac{sin\alpha}{\gamma\left(1-\frac{v}{c}\cdot cos\alpha\right)}\right)$	**4**: $\alpha' = 2\cdot arctan\left[\left(\frac{c+v}{c-v}\right)^{1/2} tan\left(\frac{\alpha}{2}\right)\right]$

α		1		2		3		4	
0	0	0	0	0	0	0	0	0	0
0,523599	30	0,869038	49,79	0,869038	49,79	0,869038	49,79	0,869038	49,79
0,785398	45	1,244669	71,31	1,244669	71,31	1,244669	71,31	1,244669	71,31
1,047198	60	1,570796	90,00	1,570796	90,00	1,570796	90,00	1,570796	90,00
1,570796	90	-1,047198	120,00	2,094395	120,00	1,047198	120,00	2,094395	120,00
2,094395	120	-0,643501	143,13	2,498092	143,13	0,643501	143,13	2,498092	143,13
2,356194	135	-0,469475	153,10	2,672117	153,10	0,469475	153,10	2,672117	153,10
2,617994	150	-0,306968	162,41	2,834625	162,41	0,306968	162,41	2,834625	162,41
3,141593	180	0	180,00	3,141593	180,00	0	180,00	3,141593	180,00

Tab. 2.4: Values for α' depending on α according to equations 1 to 4, $v = 0{,}5c$
Results presented as radian and in degrees [°] (marked grey).
Values with frame: 180°+ angle (Eq. 1) and 180°- angle (Eq. 3)

5: $\beta' = arctan\left(\frac{sin\beta}{\gamma\left(cos\beta + \frac{v}{c}\right)}\right)$	**6**: $\beta' = arccos\left(\frac{cos\beta + \frac{v}{c}}{1 + \frac{v}{c} \cdot cos\beta}\right)$
7: $\beta' = arcsin\left(\frac{sin\beta}{\gamma\left(1 + \frac{v}{c} \cdot cos\beta\right)}\right)$	**8**: $\beta' = 2 \cdot arctan\left[\left(\frac{c-v}{c+v}\right)^{1/2} tan\left(\frac{\beta}{2}\right)\right]$

β		5		6		7		8	
0	0	0	0	0	0	0	0	0	0
0,523599	30	0,306968	17,59	0,306968	17,59	0,306968	17,59	0,306968	17,59
0,785398	45	0,469475	26,90	0,469475	26,90	0,469475	26,90	0,469475	26,90
1,047198	60	0,643501	36,87	0,643501	36,87	0,643501	36,87	0,643501	36,87
1,570796	90	1,047198	60,00	1,047198	60,00	1,047198	60,00	1,047198	60,00
2,094395	120	1,570796	90,00	1,570796	90,00	1,570796	90,00	1,570796	90,00
2,356194	135	-1,244669	108,69	1,896924	108,69	1,244669	108,69	1,896924	108,69
2,617994	150	-0,869038	130,21	2,272555	130,21	0,869038	130,21	2,272555	130,21
3,141593	180	0	180	3,141593	180	0	180	3,141593	180

Tab. 2.5: Values for β' depending on β according to equations 5 to 8, $v = 0{,}5c$
Results presented as radian and in degrees [°] (marked grey).
Values with frame: 180°+ angle (Eq. 5) and 180°- angle (Eq. 7)

The considerations of equations 1 to 8 discussed so far were solely directed on the radiation angle for a light pulse, which could be measured by an observer at rest and was subsequently calculated for a moving system. In this case the angles measured in moving direction cover per definition the designation α (for the system at rest) and α' (moving) whereas β and β' are situated in opposite direction.

It was already demonstrated in chapter 2.3.2 that the investigation of the case, where the positions are changed and the moving observer is calculating values for the observer at rest, the angles evaluated by the moving observer will reveal exactly the opposite results. This means that measurements in moving direction following angle α will show the formal result of angle β' and that it will also be the same case for β and α'.

The evaluation presented so far is only valid for the equation 1. The same result will appear, however, when equation 4 is converted in a suitable way to show the value of α.
Whereas calculations for incoming signals are discussed quite often in the literature, only few solutions for outgoing signals can be found. R. Göhring [47] used the equations for outgoing signals and made a transformation to α'; this showed that the results were in accordance with the results described in the following. In the presentation by W. Rindler [28] it is defined, that the values for the velocity c shall be replaced by $-c$ and then the relevant calculations will appear. When this is done for all presented variants then it can be shown that this statement is valid for all calculations investigated here.

The results can be summarized as follows:

1: $\alpha = arctan\left(\frac{sin\alpha'}{\gamma\left(cos\alpha' + \frac{v}{c}\right)}\right)$	**2**: $\alpha = arccos\left(\frac{cos\alpha' + \frac{v}{c}}{1 + \frac{v}{c}\cdot cos\alpha'}\right)$
3: $\alpha = arcsin\left(\frac{sin\alpha'}{\gamma\left(1 + \frac{v}{c}\cdot cos\alpha'\right)}\right)$	**4**: $\alpha = 2\cdot arctan\left[\left(\frac{c-v}{c+v}\right)^{1/2} tan\left(\frac{\alpha'}{2}\right)\right]$

The same conversion is possible for the opposite case:

5: $\beta = arctan\left(\frac{sin\beta'}{\gamma\left(cos\beta' - \frac{v}{c}\right)}\right)$	**6**: $\beta = arccos\left(\frac{cos\beta' - \frac{v}{c}}{1 - \frac{v}{c}\cdot cos\beta'}\right)$
7: $\beta = arcsin\left(\frac{sin\beta'}{\gamma\left(1 - \frac{v}{c}\cdot cos\beta'\right)}\right)$	**8**: $\beta = 2\cdot arctan\left[\left(\frac{c+v}{c-v}\right)^{1/2} tan\left(\frac{\beta'}{2}\right)\right]$

Finally, it can be stated, that all presented equations are suitable for the calculation of relativistic aberration of moving observers connected to systems at rest and vice versa. The results of the aberration angles are the same for all involved participants and thus the principle of relativity is not violated. Precondition is that the effect of spatial contraction is symmetric to the middle axis of the moved body in moving direction and opposite to it.

For practical use equations 2 or 4 resp. 6 or 8 shall be preferred because they show no sinus in the formula and so no interpretation of the result is necessary for values $> 90°$. The real advantage of the geometric derivation presented here (this means equations 1 and 5) will become apparent later, when subluminal velocities of moving bodies instead of light signals will be discussed. In this case equation 1 (or 5) can be modified using a simple replacement of c by the velocity v of the second moving object, which is not possible for the other calculations. This will be especially important for discussions of questions concerning the momentum, which will be a major topic in chapter 7.

2.4 Exchange of signals in any arbitrary spatial direction

After discussion of the basic relations concerning the path of a signal in any arbitrary spatial direction, it is now possible to verify that for a signal in a moved system (here with the shape of a sphere with a standard-radius of $a = 1$) from the center to the outer shell and back, subjectively the same time will be measured compared to the system at rest. The following conditions shall be defined:

An angle α' (related to the moving direction) shall be chosen for the moved system, from which the light signal will be emitted to the outer shell. Then the following values are calculated:

1. The related angle α_1 viewed by the observer at rest,
2. The length d_1 to the outer shell,
3. The angle α_2 for the way back referring to the same angle α',

4. The length d_2 from the shell to the center,
5. The calculation of $d_T = d_1 + d_2$. The value of d_T must be exactly $2a\gamma$ to verify that the measurements in both systems (moving and at rest) are subjectively identical.

For the calculation, the equations (2.67) and (2.45) shall be used and the following relations appear:

2: $\alpha_1 = 2 \cdot arctan\left[\left(\frac{c-v}{c+v}\right)^{1/2} tan\left(\frac{\alpha'}{2}\right)\right]$	**3**: $d_1 = \frac{sin\alpha'}{sin\alpha_1}$
4: $\alpha_2 = 2 \cdot arctan\left[\left(\frac{c+v}{c-v}\right)^{1/2} tan\left(\frac{\alpha'}{2}\right)\right]$	**5**: $d_2 = \frac{sin\alpha'}{sin\alpha_2}$

In table 2.6 calculations for an example $v = 0{,}5c$ are presented. For the values $\alpha' \to 0°$ and 180° with respect to α_1 and α_2 a division of 0 by 0 would appear and it would be necessary to extrapolate, for simplification only values between 1° to 179° were selected. The values directly in moving direction and opposite to it (0° and 180°) were already determined before in chapter 2.1.

For all calculated values of d_T the result of 2γ (in this case $v = 0.5c \Rightarrow 2\gamma = 2{,}309401..$) appear. This means that in view of the observer at rest the distance travelled by the light pulse and the time needed is exactly longer by this value. All values show impressively that no deviations between the subjective measurements of the moved observer and a system at rest will appear. The time in the moving system is running slower by the calculated factor and the principle of relativity, as in all cases discussed before, will not be violated.

α' [°]	α'	α_1	α_1 [°]	d_1	α_2	α_2 [°]	d_2	d_T
1	0,017453	0,010077	0,577360	1,731963	0,030228	1,731963	0,577438	2,309401
15	0,261799	0,151727	8,693343	1,712378	0,448391	25,69090	0,597023	2,309401
30	0,523599	0,306968	17,58795	1,654701	0,869038	49,79218	0,654701	2,309401
45	0,785398	0,469475	26,89895	1,562949	1,244669	71,31426	0,746452	2,309401
60	1,047198	0,643501	36,86990	1,443376	1,570796	90	0,866025	2,309401
75	1,308997	0,834062	47,78826	1,304130	1,851500	106,0831	1,005271	2,309401
90	1,570796	1,047198	60	1,154701	2,094395	120	1,154701	2,309401
105	1,832596	1,290093	73,91689	1,005271	2,307530	132,2117	1,304130	2,309401
120	2,094395	1,570796	90	0,866025	2,498092	143,1301	1,443376	2,309401
135	2,356194	1,896924	108,6857	0,746452	2,672117	153,1010	1,562949	2,309401
150	2,617994	2,272555	130,2078	0,654701	2,834625	162,4120	1,654701	2,309401
165	2,879793	2,693202	154,3091	0,597023	2,989865	171,3067	1,712378	2,309401
179	3,124139	3,111364	178,2680	0,577438	3,131516	179,4226	1,731963	2,309401

Tab. 2.6: Calculation of values $d_T = d_1 + d_2$ according to equations 2 to 5, $v = 0.5c$
All results reveal exactly $2\gamma = 2{,}309401$