3. Lorentz-Transformation and synchronization

The calculations concerning coordinates of space and time presented so far are not sufficient for the complete understanding of the relativistic transformation procedure. Already in the year 1900 the essential additional principle of “local time” and the consequences connected with it were investigated by H. Poincaré [10]. Later A. Einstein implemented the general statement, that the local time of moving observers must always be connected by synchronization processes [12].

Inside Special Relativity the synchronization of incidents between moved observers is of paramount importance. It is part of any comprehensive lecture concerning Special Relativity, further a multitude of publications exists of which only a small part can be discussed here.

Generally, the issue can be divided in two categories:

1. The synchronization of incidents by exchanging signals,
2. The synchronization of incidents by the exchange of clocks.

The results do not correspond to the intuitive human understanding of simultaneity and are therefore not easy to understand. This is due to the fact that an exchange of signals between two observers always occurs at the speed of light, and this must be included in the considerations. In the following the connections with the synchronization of events by using signal exchange are considered first, the synchronization by means of the exchange of clocks is treated in chapter 5.

3.1 Local time and synchronization using the exchange of signals

An experimental set-up shall be discussed, where a laboratory with length a is considered as at rest and is passed by a small body with the velocity v (Fig. 3.1). On both ends named A and E of the laboratory a clock is fixed. At the first contact of the moved body at A (case a) the clock is set to the value

$$t = -\frac{a}{v} \tag{3.01}$$

When the moving body has contact at point E (case b) the clock at point A shows the value of zero. Using this procedure, the synchronization of both observers is realized. At the point zero both emitters at A and E shall send simultaneously a signal that will arrive at time

$$t = \frac{a}{c} \tag{3.02}$$

at their partners (case c).

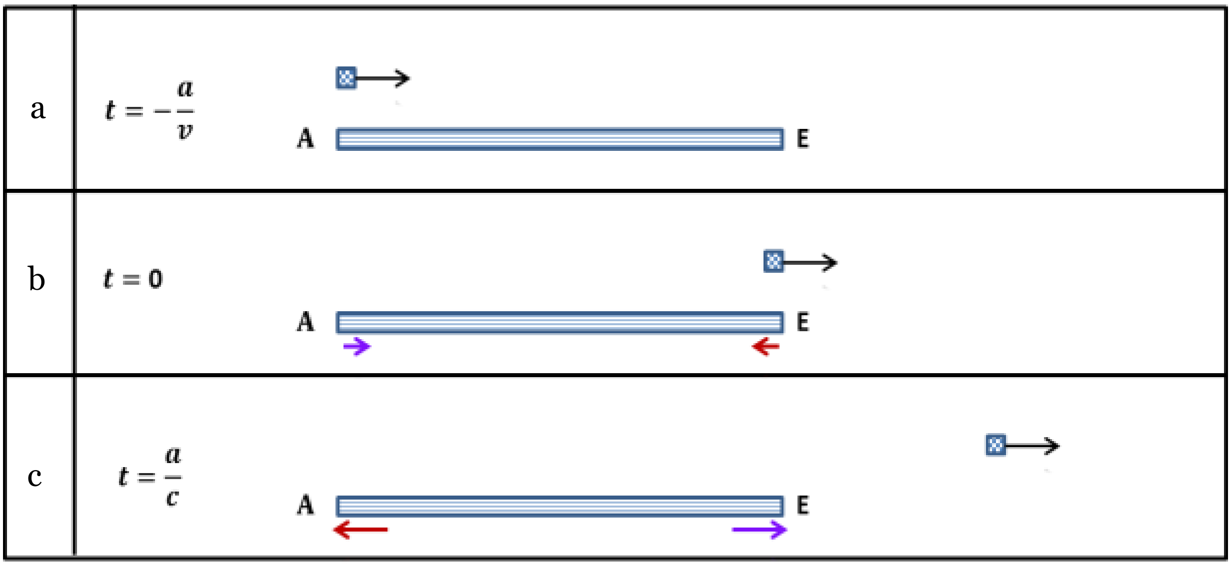

Fig. 3.1: Experimental set-up for the synchronization of an observer at rest using clocks at the ends A and E

According to the principle of relativity all participants of the experiment must find the same results, when instead of the laboratory the moving body in Fig 3.1 is considered as at rest. When these conditions are recorded a completely different diagram will appear. In Fig. 3.2 the space-time-diagram covering the new issue with the changing of the point of view is presented.

First the clock at A is passing the body at rest (presented as point A_0). Now the waiting time is starting; for the observer at rest the time dilatation must be considered. The clock in the position E is passing the body at rest at E_1 (the presentation is respecting the fact, that the moving laboratory is shortened by the factor γ because of its movement). At that point a signal is send to A which will be received there at time A_4. After the end of the waiting time A will send at time A_2 also a signal to E which will be received there at time E_3.

It is clearly visible, that from the point of view of the observer at rest the times for the moved laboratory at A and E are not identical to his observations. In this case the time zero is depending on the distance to the observer at rest and follows a line which is marked as x' in the diagram.

Generally, this is one of the most important features of Special Relativity. This effect is commonly called "Relativity of Simultaneity".

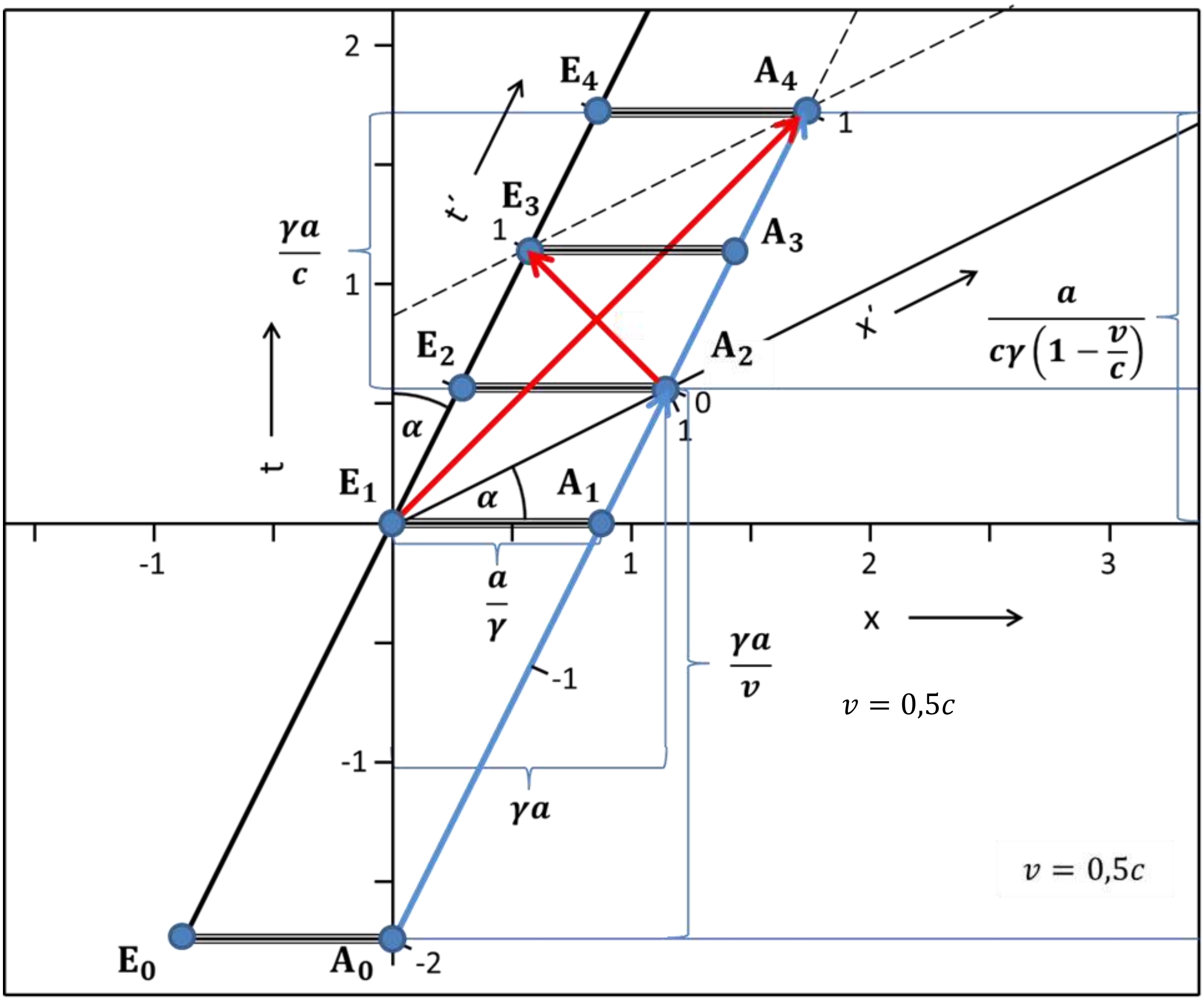

Fig. 3.2 Experimental set-up for the synchronization of a moving observer using clocks at the ends A and E.

The synchronization difference Δt_S can be determined easily using

$$\Delta t_S = \frac{a}{c\gamma\left(1 - \frac{v}{c}\right)} - \frac{\gamma a}{c} = \frac{\gamma a}{c}\left(1 + \frac{v}{c}\right) - \frac{\gamma a}{c} \tag{3.03}$$

$$\Delta t_S = \frac{\gamma a v}{c^2} \tag{3.04}$$

The angle between the x'- and the x-axis is calculated from the synchronization difference divided by γa

$$\tan\alpha = \frac{c \cdot \Delta t_S}{\gamma a} = \frac{v}{c} \tag{3.05}$$

and is thus identical with the angle between the ct'- and ct-axes.

The diagram developed here has interesting features, which will be discussed in the following.

3.2 Minkowski-diagram

The diagram presented above was introduced into Special Relativity by Hermann Minkowski (1864-1909) who, among many important scientific contributions, developed this presentation later named after him [15c].

Minkowski diagrams show several peculiarities. First of all, usually not the representation of t but of ct over x is chosen. This gives both axes the same dimension (length) and direct derivations can be made from them. After normalization, the appearance shown in Fig. 3.3 is obtained. In this form, the diagram shows a mirror symmetry with respect to the 45° axis passing through the origin.

It is possible to determine directly from these diagrams the coordinates which result for the stationary (x, ct) and for the moving observer (x', ct') for the same circumstances. In the diagram Fig. 3.4 the point $P_{x,ct}$ with the coordinates $x = 3$ and $ct = 2$ is shown as an example. This is the value, at which a moving observer from the view of the stationary system is at a distance of 3 length units (LU) after 2 time units (TU) referred to the origin.

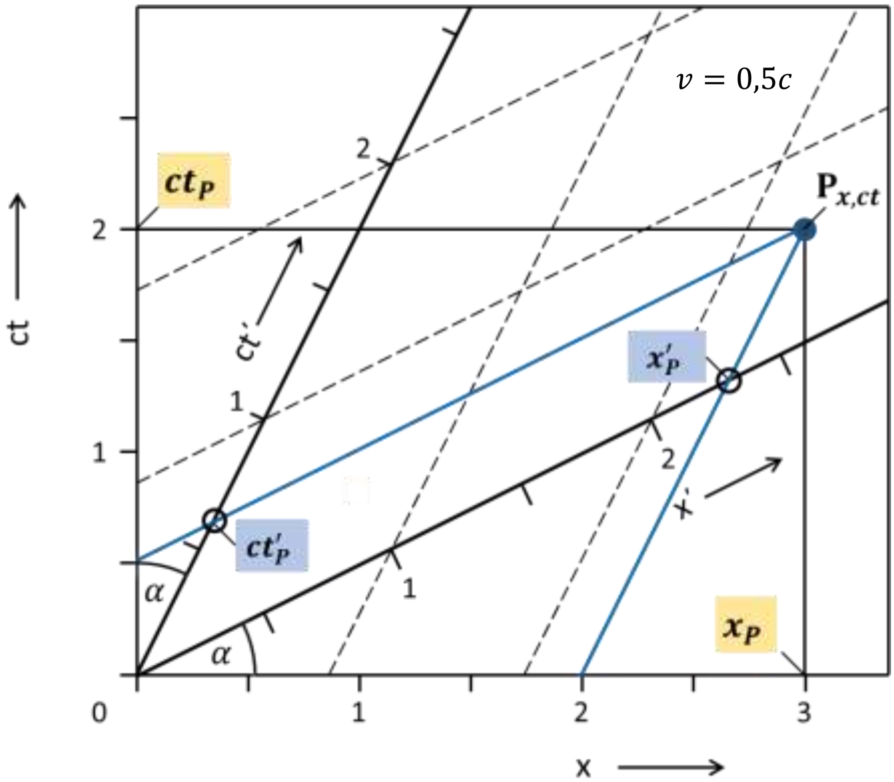

Fig. 3.3: Minkowski diagram: Example with point $x = 3$ and $ct = 2$. Graphical determination of the coordinates in the moving system (x', ct') '.

The x', ct' − coordinate system is not rectangular but has angles α to the system x, ct. Therefore the coordinates are also read under this angle. Parallels to the x' and ct' axis are formed. The values for x'_P and ct'_P can then be read from the intersections with the axes $ct' = 0$ and $x' = 0$ respectively as shown.

It will be shown in the next chapter that a purely graphical/geometric derivation leads in consequence to the Lorentz transformation equations. This is absolutely necessary, because otherwise there would be contradictions within the theory.

3.3 Lorentz-Transformation

For the derivation of the Lorentz transformation there is a multiplicity of approaches, which can be mentioned here only exemplarily. According to the classification introduced by M. Born [26] and still used today [47], there is basically the graphical and the algebraic approach. While the graphical derivation is rarely used [e.g. 26a], there is a multitude of variants for the algebraic approach. These range from the classical representation [12,29] to the "fastest" derivation [30], conventional approaches [31,32] and to the use of the tensor calculus [27,28,33]. Moreover, parts of the graphical and algebraic derivation can also be combined [19]. Since the Lorentz transformation is one of the most important elements of Special Relativity, its derivation will be shown here with selected examples for both basic approaches.

In principle, the present relations must be linear. If there were e.g. quadratic terms, then derivations after space or time would depend on the space or the time itself. All physical laws, which contain derivations after place or time (e.g. velocity, accelerations) would then depend on the zero point of the corresponding space or time scale in case of non-linear relations. In such a case, however, this could be the subject of direct measurements and thus contradicts the general idea of the homogeneity of space and time. A further point is that the relations to be determined in the limit case of small velocities must pass over into the Galilei transformation of the classical mechanics.

In the following, first a graphical (and geometric) derivation of the Lorentz transformation from the Minkowski diagram is presented. In contrast to the approach of M. Born [26a], which works with proportion relations and the Pythagorean theorem, angular functions and geometrical approaches are used here and a particularly clear representation appears. Subsequently, a selected algebraic approach is presented.

At this stage, an important point shall be briefly discussed. According to the principle of the constancy of the speed of light in all inertial systems, measurements of the speed of light will lead to the same result for the reference system ("resting") and for an observer moving relative to it (chapter 1.6). This is subjectively correct. However, the derivations discussed in the following are based *exclusively* on the speed of light of the *reference system* and thus describe the observations made from this, from which finally the Lorentz transformations are resulting.

3.3.1 Derivation of the Lorentz-Transformation using the Minkowski diagram

As was already explained, the representation of the Minkowski diagram can be derived exclusively using time dilation, space contraction and synchronization difference. Beyond that only the assumption of the isotropy of time and space as well as the constancy of the speed of light (in the system at rest) is necessary. In the following it will be shown that at the transition between the represented systems of this diagram, relations corresponding to the Lorentz transformation must inevitably result.

When an arbitrary point $P_{x,ct}$ is considered in this diagram (Fig. 3.4), the coordinates can be calculated with the help of the values marked in yellow.

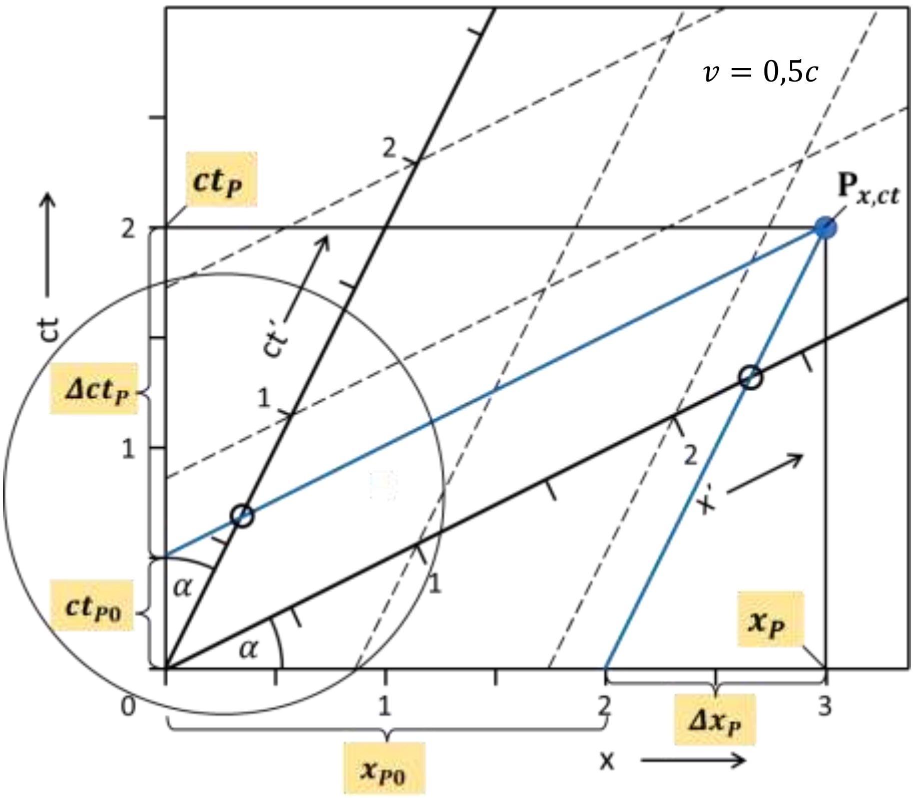

Fig. 3.4: Minkowski diagram with coordinate determination of point $P_{x,ct}$ in the moving system. Quantities relevant for the calculation are colored yellow.

First, parallels to the x' and ct' axes are formed and their intersections with the ct/x-coordinate system are determined. The resulting values ct_{P0} and x_{P0} can be converted into x'_P and ct'_P. For this purpose, an intermediate calculation is required in the range around 1. For this purpose, a circle is drawn in Fig. 3.4, the contents of which are shown in higher resolution in Fig. 3.5.

In this diagram Fig. 3.5 all values are normalized to 1. In the case shown, no change of location occurs within the moving laboratory, i.e. the movement takes place on the ct'-axis. Then, as already shown in chapter 2, the dependence $d = \gamma \cdot ct_1$ applies for the case $ct = 1$. It follows

$$\tan\alpha = \frac{v}{c} = \frac{b}{d} = \frac{e}{b} \tag{3.10}$$

and from this

$$e = d\frac{v^2}{c^2} \tag{3.11}$$

Because of $f = d - e$, it follows after substituting eq. (3.11)

$$f = d - d\frac{v^2}{c^2} = d\left(1 - \frac{v^2}{c^2}\right) = \frac{d}{\gamma^2} = \frac{ct_1}{\gamma} \tag{3.12}$$

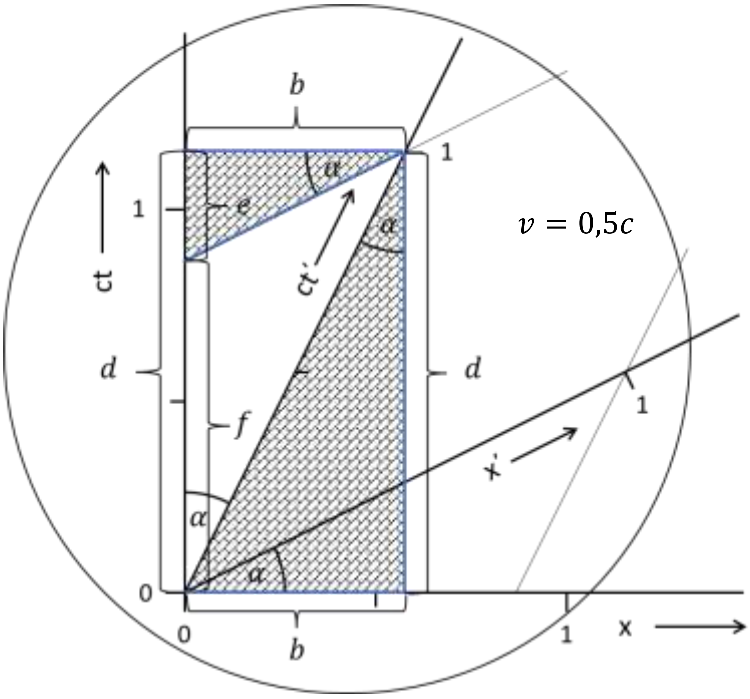

Fig. 3.5: Detail from Fig. 3.4, determination of f corresponding to ct_{P0} from Fig. 3.4.

For the x'-axis, the same relationship applies for symmetry reasons. It follows first for the value ct'_P:

$$ct'_P = \gamma \cdot ct_{P0} \tag{3.13}$$

From the geometrical conditions in Fig. 3.4, we get

$$ct'_P = \gamma\,(ct_P - \Delta ct_P) \tag{3.14}$$

Because of

$$\tan\alpha = \frac{\Delta ct_P}{x_P} = \frac{v}{c} \tag{3.15}$$

then finally appears

$$t'_P = \gamma\left(t_P - \frac{v}{c^2}x_P\right) \tag{3.16}$$

For x'_P we obtain in the same way

$$x'_P = \gamma \cdot x_{P0} \tag{3.17}$$

$$x'_P = \gamma\,(x_P - \Delta x_P) \tag{3.18}$$

$$\tan\alpha = \frac{\Delta x_P}{ct_P} = \frac{v}{c} \tag{3.19}$$

$$x'_P = \gamma\,(x_P - v\,t_P) \tag{3.20}$$

The calculation results in the following values

System at rest	Moved System
$x_P = 3$	$x'_P = 2{,}309$
$ct_P = 2$	$ct'_P = 0{,}577$

The equations (3.16) and (3.20) correspond exactly to the relations of the Lorentz transformation as they were already presented in Eq. (1.01) and (1.02). Thus it is shown that these equations can be derived from a Minkowski diagram by establishing simple geometrical correlations.

3.3.2 Algebraic concept for the derivation of the Lorentz-Transformation

To complete the considerations concerning the Lorentz-Transformation in addition a "classic" approach, which means a typical derivation of the equations used in the literature, shall be discussed. To show this concept in detail the presentation of H. J. Lüdde and T. Rühl [34] was chosen, because it has a basic approach and does not need assumptions during the derivation, which show later that they are reasonable. A similar derivation was also used by A. Einstein in the year 1905, although his only comment was "after easy calculation" without showing any details [12b].

Using this concept, two systems shall be looked at which are moving against each other. It is generally required that these are inertial systems, which means acceleration and rotation is not permitted. The position of any point in these systems is characterized by three coordinates for the space and one for the time. For the system S these are x, y, z, t and S′ with $x', y, 'z', t'$. It is assumed, that the systems move against each other with a speed of v concerning the x- coordinate and that in y- and z- direction no motion exists.

First the situation is discussed that the point of origin (where space and time are defined as zero) of both systems get in contact at the time

$$t = t' = 0 \tag{3.40}$$

In this case the correlations between the coordinates are, because of the required linearity

$$x' = Ax + Bt, \qquad y' = y, \qquad z' = z, \qquad t' = Cx + Dt \tag{3.41}$$

This means that t is no longer invariant concerning space and furthermore x is not invariant concerning time. Thus, for an arbitrary sphere with a light emitter in the center the following equations will apply:

$$S: \quad x^2 + y^2 + z^2 = c^2t^2 \tag{3.42}$$

$$S': \quad x'^2 + y'^2 + z'^2 = c^2t'^2 \tag{3.43}$$

Hence

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2 \tag{3.44}$$

For the solution of the equations first the system-velocities are considered. In view of system S' the velocity of S is

$$v = \frac{x}{t} \tag{3.45}$$

When the situation is discussed that both systems have contact in the point of origin Eq. (3.41) develops to

$$0 = Avt + Bt \tag{3.46}$$

or

$$B = -Av \tag{3.47}$$

The use of Eq. (3.44) leads to

$$(Ax + Bt)^2 - c^2(Cx + Dt)^2 = x^2 - c^2t^2 \tag{3.48}$$

and

$$x^2(A^2 - c^2D^2 - 1) + 2xt(AB - c^2CD) + t^2(B^2 - c^2D^2 + c^2) = 0 \tag{3.49}$$

Because the relations (3.48) and (3.49) are valid for arbitrary values of space and time the following equations apply:

$$A^2 - c^2C^2 - 1 = 0 \tag{3.50}$$

$$AB - c^2CD = 0 \tag{3.51}$$

$$B^2 - c^2D^2 + c^2 = 0 \tag{3.52}$$

The solution of this system with 4 equations and 4 unknown factors [Eq. (3.47) and also Eq. (3.50) - (3.52)] leads to the following relations

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{3.53}$$

$$x' = \gamma(x - vt) \tag{3.54}$$

The y- and z- coordinates remain unchanged.

The results of the derivation presented here are in full agreement with the Lorentz-Transformation already discussed before several times. The requirements concerning time dilatation, space contraction and local time (with asynchronous characteristics) can be derived out of subsequent calculations. This contrasts with the calculations presented before, where the equations were derived using a graphic approach; in this case time dilation and length contraction were preconditions and not the results of calculations.

Finally the question remains, what significance the result has for the interpretation of the conditions. In chapter 2.2 it was already presented in detail that it is impossible for an observer at rest or in a moving system using the exchange of signals to decide about the state of movement. This is caused by the simultaneously appearing effects of dilatation of time and contraction of space.

However, it is by no means the case that an observer at rest is determining a different speed of light in the moving system; in his view the speed of light of his system will be valid for all investigations instead. The fact that the moving observer will find the same results in comparison to the system at rest is exclusively caused by differences in the synchronization procedures between the two systems. This question will be taken up again in chapter 11.

3.4 Einstein-synchronization

The synchronization procedure later named after Albert Einstein was first mentioned in his pioneering publication in the year 1905 [12]. To illustrate this point further, an extract of the original work is presented in Fig. 3.5, which was part of the derivation of the Lorentz-Transformation. The following equation is of special interest

$$\frac{1}{2}(\tau_0 + \tau_2) = \tau_1 \tag{3.60}$$

Einstein used Greek letters for the time in a moving system, for which today generally t' is taken (further he used the letter V, not c for the speed of light); today the equation is generally presented in a different form like

$$\frac{1}{2}(t_0' + t_2') = t_1' \tag{3.61}$$

It is a special characteristic of this equation, that the synchronization is solely depending on the exchange of signals between the participants.

The synchronization procedure following this specification can generally be characterized as follows:

> Clock U(0) is situated in the coordinate origin of an arbitrary inertial system. An identical clock U(x) is located at a different point with the distance x. When U(0) is showing time t_0 a light signal is emitted from here to point x and from there immediately reflected to the coordinate origin. At arrival U(0) is showing time t_2. U(x) is synchronized with U(0) when U(x) at the time of reflection is showing time t_1 following the relation:

$$t_1 = t_0 + \frac{1}{2}(t_2 - t_0) \tag{3.62}$$

Equation Eq. (3.62) is identical to Eq (3.60) resp. (3.61). This is independent from the situation, whether the clocks are at rest or shall be moved (which means the use of t or t' is possible).

To any system of values x, y, z, t, which completely defines the place and time of an event in a stationary system, a system of values ξ, η, ζ, τ, determining that event relatively to the system k belongs to it, and the task is now to find a system of equations connecting these variables.

First it is clear that the equations must be linear on account of the properties of homogeneity which we attribute to space and time.

If we set $x' = x - vt$, it is clear that a point at rest in the system k must belong to a system of values x', y, z, independent of time. We first determine τ as a function of x', y, z, and t. To do this we have to express in equations that τ is nothing else than the summation of the reading of clocks at rest in system k, which have been synchronized according to the rules given in § 1.

From the origin of system k let a ray be emitted at time τ_0 along the X-axis to x', and at time τ_1 be reflected to the origin of the coordinates, arriving there at time τ_2, then we will find

$$\frac{1}{2}(\tau_0 + \tau_2) = \tau_1$$

or, by inserting the arguments of the function τ and applying the principle of the constancy of the speed of light in the stationary system:

$$\frac{1}{2}\left[\tau(0,0,0,t) + \tau\left(0,0,0,\left\{t + \frac{x'}{V-x} + \frac{x'}{V+x}\right\}\right)\right]$$
$$= \tau\left(x', 0,0, t + \frac{x'}{V-x}\right)$$

Hence, if x' is chosen infinitesimally small

$$\frac{1}{2}\left(\frac{1}{V-x} + \frac{1}{V+x}\right)\frac{\partial\tau}{\partial t} = \frac{\partial\tau}{\partial x'} + \frac{1}{V-v}\frac{\partial\tau}{\partial t}$$

or

$$\frac{\partial\tau}{\partial x'} + \frac{v}{V^2 - v^2}\frac{\partial\tau}{\partial t} = 0$$

It shall be noted that it is possible to choose any other point of origin for the coordinates of the ray, and the equation just obtained is therefore valid for all values of x', y, z.

Fig. 3.5: Extract from original publication of Albert Einstein [12a], translated

The definition used in these equations is not giving information, whether synchronization is still valid at a later point in time or not. In principle the following situations are possible:

a) U(x) remains stationary in relation to U(0),

b) U(x) is passing U(0) in short distance to be synchronized and then moving away,

c) U(x) is passing U(0) in a long distance without direct contact.

It is immediately clear for situation a) that the factor γ is always identical for both clocks and so the synchronization can be repeated without difference at any time. Situations b) and c) were dealt with in chapters 2.1.1 resp. 2.1.2. In both cases it was shown, that independent from the distance of objects no differences of their observations are detectable. The only precondition is, that the Lorentz-Transformation is taken as a basis.

Exact interpretation of the situation makes clear, that when using hypothetical superluminal velocities sending information to an observer, differences would appear. However, according to the assumptions made, this is not possible and so synchronization differences cannot occur. As already discussed, the appearing situation is called “Relativity of Simultaneity”.

Current concepts for derivation of the Lorentz-Equations generally avoid using the form Einstein selected in the year 1905. In a normal case a presentation using equations Eq. (3.42) and Eq. (3.43) is taken (which was used as a basis for calculation in chapter 3.3.2)

$$S: \quad x^2 + y^2 + z^2 = c^2 t^2 \tag{3.42}$$

$$S': \quad x'^2 + y'^2 + z'^2 = c^2 t'^2 \tag{3.43}$$

The equation system can be interpreted in a way, that the transition from Eq. (3.42) to Eq. (3.43) is in accordance with Einstein synchronization and this relation is implicitly included. Einstein himself in his book about the theory of relativity written as a “simple version” [29], first edited in the year 1916, also used a similar approach. Obviously, he also shared the opinion that this would be easier to understand.

The Einstein-synchronization, connected with Eq. (3.62), is a definition, not an observation. The Einstein synchronization is of paramount importance for the Theory of Special Relativity and is widely discussed until today [19,20,35]. After the presentation of additional important aspects, it will be discussed again in more detail in this investigation (chapter 11.2).

.