6. Relations for mass, momentum, force, and energy

In this chapter results connected with the relativistic mass increase will be presented. First the well-known effect on the kinetic energy will be discussed, followed by some new investigations. These are the “spring paradox”, the relativistic consideration of the elastic collision (important for the examination of collisions of elementary particles), the exchange of signals during and after acceleration and the concept of a relativistic rocket equation. Because some of the delineations show no approach to an analytical solution, numerical evaluation concepts combined with examples for calculations are added in separate files for these cases.

None of the examinations show any contradictions to the Lorentz Transformation or the basic principles of relativity.

6.1 Relativistic mass increase and energy

During the historical development of the investigations concerning relativistic mass, it was first realized that there are differences between a “longitudinal” and “transversal” mass increase for high velocities. These terms were introduced by H. A. Lorentz [13,42], because during the acceleration of electrons differences were measured depending on their movement. According to experiments the transversal mass m_t and the longitudinal mass m_l showed the following values:

$$m_t = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{6.01}$$

$$m_l = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \tag{6.02}$$

During these experiments the mass was measured in a way, that the acting force was divided by the acceleration using Newton´s law

$$m = \frac{F}{a} \tag{6.03}$$

The transverse acceleration is leading to a constant circular motion, while a longitudinal acceleration is increasing the velocity of the object and therefore both the longitudinal and transverse mass of the body is raised.

According today´s standard of knowledge the equation (6.01) is presenting the correct increase of mass during acceleration, whereas Eq. (6.02) is derived, when instead of Eq. (6.03) the complete notation of Newton´s formula for the force is used

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}(mv)}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t} v + m \frac{\mathrm{d}v}{\mathrm{d}t} \tag{6.04}$$

If Eq. (6.01) is combined with Eq. (6.04) then

$$F = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\right) v + \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \frac{\mathrm{d}v}{\mathrm{d}t} \tag{6.05}$$

With

$$\frac{\mathrm{d}m}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}m}{\mathrm{d}v} \cdot \frac{\mathrm{d}v}{\mathrm{d}t} \tag{6.06}$$

the equation develops to

$$F = \left(-\frac{1}{2}\right)\left(\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right)\left(-2\frac{v}{c^2}\right)\frac{\mathrm{d}v}{\mathrm{d}t} v + \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \frac{\mathrm{d}v}{\mathrm{d}t}$$

$$F = \left(\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\right)\left(\frac{v^2}{c^2}\right)\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{m_0\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \frac{\mathrm{d}v}{\mathrm{d}t} \tag{6.07}$$

and thus, the value in Eq. (6.02) for the longitudinal mass is the result. So, the equations for the different masses are identical and therefore since the mid of the 20th century the separation was cancelled and today normally the general term “relativistic mass” according to Eq. (6.01) is used.

It is apparent that equation Eq. (6.07) can be directly transformed to

$$F = \frac{m_0}{\gamma^3} a \tag{6.08}$$

This means that for a constant force acting from the *system at rest*, the acceleration occurring in the moving system (also measured from the system at rest) differs by a factor γ^3. This law was derived by H. A. Lorentz for an electric field acting on an electron from the outside. When considering accelerations caused by effects within a moving system (such as valid for a rocket engine), the same laws apply. As shown in chapter 6.4, the factor γ^3 results also if the relativistic velocity addition is chosen as the only criterion for derivation.

In the following the kinetic energy of a body in motion shall be discussed. To realize this, the relativistic (longitudinal) mass according to (6.07) is considered, because this is the complete equation that describes an increase of the velocity. The force which is necessary to accelerate a mass is therefore defined as

$$F = \frac{m_0 \cdot a}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \tag{6.09}$$

The necessary acceleration energy is

$$W_{1,2} = \int_{v_1}^{v_2} F \cdot \mathrm{d}s = \int_{v_1}^{v_2} \frac{m_0 \cdot a}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \cdot \mathrm{d}s = \int_{v_1}^{v_2} \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \cdot \frac{\mathrm{d}v}{\mathrm{d}t} \mathrm{d}s \tag{6.10}$$

Because of

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \tag{6.11}$$

it applies

$$W_{1,2} = \int_{v_1}^{v_2} \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} v \, \mathrm{d}v \tag{6.12}$$

and finally

$$W_{1,2} = \left. \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \right|_{v_1}^{v_2} \tag{6.13}$$

For $v_1 = 0$ and $v_2 = v$ follows

$$W = E_{kin} = m_0 c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) = m_0 c^2 (\gamma - 1) \tag{6.14}$$

The Taylor expansion of the square root leads to

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{1 \cdot 3}{2 \cdot 4} \frac{v^4}{c^4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{v^6}{c^6} + + \cdots \tag{6.15}$$

and for $v \ll c$ the classical formula for the kinetic energy is derived

$$E_{kin} \cong \frac{m_0}{2} v^2 \tag{6.16}$$

The equation (6.14) was developed by A. Einstein already in the year 1905 [22]. It contains implicit the first consideration of the equivalence of mass and energy and leads generally to

$$E = mc^2 \tag{6.17}$$

This is most probably the best-known formula in modern physics.

6.2 Spring paradox

In the following the situation shall be discussed, in which way a simple spiral spring and a mass attached to it will behave, when different experiments in a system at rest and in motion will be performed. To realize this at first 3 different experimental arrangements will be examined and in a second step the correlations for the energy are investigated and finally assessed.

6.2.1 Simple elongation of a spring

The simplest way to realize a static displacement of a spiral spring (this means without oscillation) is straining using a weight. This procedure is not suitable for a discussion using Special Relativity, however, because the value of the displacement is defined by the gravitational constant and thus by the mass of the earth. It is therefore not possible to carry out an undisputed examination. In this case a concept using General Relativity would be necessary.

Because of this reason a different technique for the generation of a displacement is necessary. For realization, the straining with a repulsive force is chosen, when caused by steadily flowing gas a constant force will be applied to the spring. Thus, the spring constant k can be derived by

$$F = k \cdot s \tag{6.20}$$

In this case F is the norm of the generated force and s of the displacement. When this experimental set-up is transferred into motion and the elongation of the spring is in a position transverse to the system at rest, the observers at rest and in motion must detect the same displacement of the spring because the “principle of identity” is valid. For the observer in motion the spring constant must be the same as in the case discussed before. The observer at rest will, however, because of time dilatation and increasing of the relativistic mass, realize the following differences:

1. The number of gas-molecules per time unit generating the repulsion force is reduced by the factor γ.
2. The mass of any single molecule of the gas is increased by the factor γ.
3. The velocity of the gas molecules moving in transverse direction (in relation to the observed direction of motion) is reduced by the factor γ.

It must be added to point 3 that the total speed of a flowing gas molecule is exactly the same compared to the situation for an experiment at rest. The reason for this is that the way is increasing by the factor γ but the angle of the gas flow is different by the factor $\alpha = arctan\, v/c$ to the transverse direction. This is the same situation why a light beam is travelling a longer way to a target in transverse direction in view of an observer at rest. The transverse component of the velocity is not affected by this, however, and is therefore reduced by the factor γ. These relations must be valid to make sure, that the moved observer is realizing the same situation compared to an observer at rest. In summary the considerations lead to the equation

$$k = \gamma \cdot k' \tag{6.21}$$

This means, that the spring constant in the system in motion is lower by the factor γ when it is monitored by the observer in a system at rest. This fact, which is surprising at first sight, is necessary to make sure that no discrepancies with other experimental configurations appear. This will be shown in the following.

6.2.2 Rotation

Instead of using a repulsion force the displacement of a spring can also be generated by its existing torsion characteristics. First in a system at rest the value for the peripheral velocity depending on the dislocation of the spring and so the existing force is determined. When this set-up is accelerated to a higher velocity and the experiment is repeated (using again the orientation transverse to motion) the following value for the centrifugal force is calculated

$$F_z' = m' \cdot \frac{v'^2}{r} = \frac{F_z}{\gamma} \tag{6.22}$$

Reason for the difference to the system at rest is the fact that the peripheral velocity v is occurring in a quadratic form in this equation. The relation is valid because the speed is slower in view of the observer at rest and the mass m is increasing in the discussed manner.

6.2.3 Harmonic oscillation

A similar situation is observed when the spring is performing an oscillation. In this case the following differential equation is valid

$$\ddot{x} + \omega_0^2 \cdot x = 0 \tag{6.23}$$

with

$$\omega_0^2 = \frac{k}{m} \tag{6.24}$$

and

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} \tag{6.25}$$

where ω_0 is the angular frequency and T_0 the oscillation time. When this experimental set-up is accelerated to a higher speed (again transverse to the direction of motion) the amplitude will be reduced by the factor γ. This leads to the following relation

$$T_0' = 2\pi\sqrt{\frac{m'}{k'}} = 2\pi\sqrt{\frac{\gamma^2 \cdot m}{k}} = \gamma \cdot T_0 \tag{6.26}$$

In this case also a reduction of the spring constant is necessary to avoid discrepancies with the principle of relativity.

6.2.4 Literature survey

In the literature no variants of these experiments are discussed (at least not known by the author). There is, however, an additional interpretation of the experiment with a “broken lever” (first discussed by G. N. Lewis and R. C. Tolman), which is a variant of the Trouton-

Noble Experiment, were a similar situation is discussed by P. S. Epstein [43]. Based on the general approach by A. Sommerfeld [44] the following relations were developed

$$f_x = f_x' \qquad f_y = \frac{f_y'}{\gamma} \tag{6.27}$$

where f_x and f_y are the components of the "Newtonian force". This description explains the relations developed for springs like the decrease of the force in transverse direction by an observer at rest.

6.2.5 Considerations of energy

Due to these relations a further effect appears, however, which is leading to an apparent contradiction. Considering the internal energy of the spring

$$E_{pot} = \int_0^s F(s)ds = \int_0^s k \cdot s\, ds \tag{6.28}$$

it is obviously clear, that during straining the energy is depending on the force resp. on the spring constant in a linear relationship. Assessing the examples discussed before this would mean, that the mechanical energy of a spring is decreasing with higher velocities. This is clearly a violation of the universal principle of conservation of energy. If a strained spring is accelerated and then released an observer at rest would measure a lower energy compared to the value which was necessary when loading the spring. Looking the other way round the spring would have a higher internal energy after a deceleration.

To dissolve the apparent paradox first an additional examination of the total energy shall be carried out. For this purpose, the total energy of a mass is observed which is moving with a velocity v_1. This situation is according to the equation established in chapter. 6.1

$$E_1 = \gamma_1 m_0 c^2 \tag{6.29}$$

Now the case is investigated, that the mass is moving in a direction transverse to this (relative to the observer at rest), with a speed of v_2 measured by the observer in motion. The observer at rest will find a reduced value of

$$v_2' = \frac{v_2}{\gamma_1} \tag{6.30}$$

because of time dilatation. According to the relativistic addition of velocities (see chapter 4.1, Eq. (4.20) with $\alpha = 90°$) this will lead to

$$v_T = \sqrt{\left(\frac{v_1}{c}\right)^2 + \left(\frac{v_2}{\gamma_1 c}\right)^2 - \left(\frac{v_1 v_2}{\gamma_1 c^2}\right)^2} \tag{6.31}$$

The energy of this mass is

$$E_T = \gamma_T m_0 c^2 \tag{6.32}$$

The differences of these energies are

$$\Delta E = \gamma_T m_0 c^2 - \gamma_1 m_0 c^2 \tag{6.33}$$

with

$$\Delta E = \frac{m_0 c^2}{\sqrt{1-\left(\frac{v_T}{c}\right)^2}} - \frac{m_0 c^2}{\sqrt{1-\left(\frac{v_1}{c}\right)^2}} \tag{6.34}$$

A Taylor expansion using $v_1, v_2 \ll c$ for this equation and the insertion of v_T according to Eq (6.31) leads to the value

$$\Delta E \cong \left[1+\frac{1}{2}\left(\left(\frac{v_1}{c}\right)^2+\left(\frac{v_2}{\gamma_1 c}\right)^2-\left(\frac{v_1 v_2}{\gamma_1 c^2}\right)^2\right)-\left(1+\frac{1}{2}\left(\frac{v_1}{c}\right)^2\right)\right] m_0 c^2$$

$$= \frac{1}{2}\left[\left(\frac{v_2}{\gamma_1 c}\right)^2-\left(\frac{v_1 v_2}{\gamma_1 c^2}\right)^2\right] m_0 c^2$$

$$= \frac{1}{2}\left[\left(\frac{v_2}{\gamma_1 c}\right)^2\left(1-\left(\frac{v_1}{c}\right)^2\right)\right] m_0 c^2 = \frac{1}{2} m_0 v_2^2 \tag{6.35}$$

This is exactly the relation for the kinetic energy of a body in motion for nonrelativistic condition and shows that the balance of energy is obeyed in this case. The discrepancies concerning the energy of a spring are generated by the fact, that the force is a physical value with a direction. In this case the strange situation occurs that force and acceleration having different orientations. This issue was already discovered by P. S. Epstein in the year 1911 [43]. Although in this paper - according to the knowledge at that time - the mass was assigned the character of a tensor and the relationships discussed in chapter 6.1 for the force in moving direction and transverse to it where unknown, this is the solution to solve the discrepancies of the paradox.

6.3 Relativistic elastic collision

A further non-linear examination is possible for relativistic elastic collision. This will not be of importance when macroscopic observers are considered, because velocities to create a noticeable effect would certainly destroy the participating bodies on impact. However, when the effect on the behavior of elementary particles is examined, e.g. in particle colliders, it is an interesting question, how the tracking of the reaction changes when it is viewed by observers with different velocities relative to the experimental set-up.

The foundation for the calculation is – like for the non-relativistic examination – the laws of conservation for energy and momentum. The relevant relations for momentum and energy are

Rel. momentum: $$\vec{p} = \gamma m \vec{v} \tag{6.40}$$

Rel. kinetic energy: $$E = (\gamma - 1) m c^2 \tag{6.41}$$

When in a simple example it is assumed that 2 masses are colliding centrally without deviation, then for the momentum the presentation as vector can be skipped and the conservation laws are

$$m_1\gamma_1 v_1 + m_2\gamma_2 v_2 = m_1\gamma_3 v_3 + m_2\gamma_4 v_4 \tag{6.42}$$

$$(\gamma_1 - 1)m_1c^2 + (\gamma_2 - 1)m_2c^2 = (\gamma_3 - 1)m_1c^2 + (\gamma_4 - 1)m_1c^2 \tag{6.43}$$

where v_1 and v_2 are the velocities before and v_3 and v_4 after collision. This leads to

$$p = m_1\gamma_1 v_1 + m_2\gamma_2 v_2 = m_1\gamma_3 v_3 + m_2\gamma_4 v_4 \tag{6.44}$$

and

$$\frac{E_0}{c^2} = (\gamma_1 - 1)m_1 + (\gamma_2 - 1)m_2 = (\gamma_3 - 1)m_1 + (\gamma_4 - 1)m_2 \tag{6.45}$$

The determination of the results for v_3 and v_4 is not possible in closed analytical form and so for the solution a numerical approach is necessary. For the required calculation the principle of bisection is used. An example for the required computation is presented in annex A in the attachment.

For the examination of the non-relativistic case the equation for the momentum in Eq. (6.44) is modified

$$m_1 v_1 + m_2 v_2 = m_1 v_3 + m_2 v_4 \tag{6.46}$$

where simply the values for γ are skipped, and further the use of the approximation formula

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots \tag{6.47}$$

for $v \ll c$ and insertion into Eq. (6.45) leads to

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_3^2 + \frac{1}{2}m_2 v_4^2 \tag{6.48}$$

When Eq. (6.46) and Eq. (6.48) are suitably transformed it applies

$$m_1(v_1 - v_3) = m_2(v_4 - v_2) \tag{6.49}$$

and

$$m_1(v_1 - v_3)(v_1 + v_3) = m_2(v_4 - v_2)(v_4 + v_2) \tag{6.50}$$

Hence, after division of both equations

$$v_1 + v_3 = v_4 + v_2 \tag{6.51}$$

and after insertion in Eq. (6.49) the classical equations for the central collision can be derived in a simple way

$$v_3 = 2\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} - v_1 \tag{6.52}$$

and

$$v_4 = 2\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} - v_2 \tag{6.53}$$

It is obvious that the result represents a simple analytical solution and that for this case no numerical calculations are necessary.

Still open is the question, how the results will be tracked by observers with different velocities relative to the collision. To examine this, the circumstances for the situation before and after collision must be considered in detail. In annex A the calculation of the values of v_3 and v_4 is presented first, furthermore the equations for the relativistic addition of velocities according to the following relations are calculated, which is then subject to further comparison:

$$v_T(v_1, v_2) = \frac{v_1 - v_2}{1 - \frac{v_1 v_2}{c^2}} \tag{6.54}$$

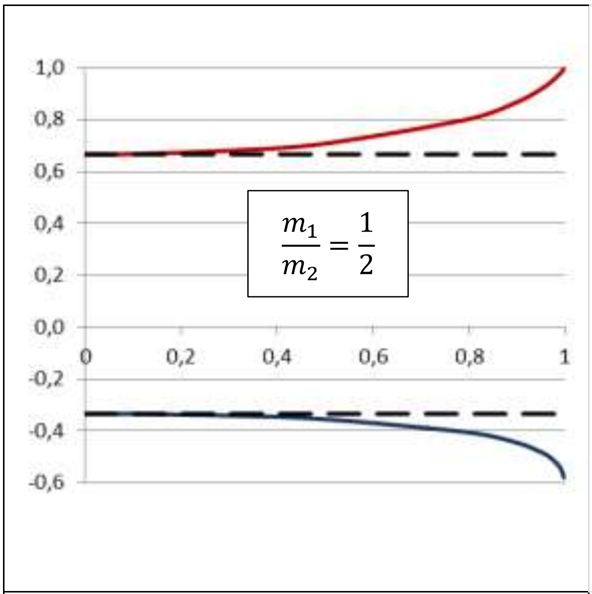
$$v_T(v_4, v_3) = \frac{v_4 - v_3}{1 - \frac{v_4 v_3}{c^2}} \tag{6.55}$$

For a meaningful comparison between both results the quotient will be calculated first and then, because of the small deviation, the appearing value will be subtracted by1 resulting the error range

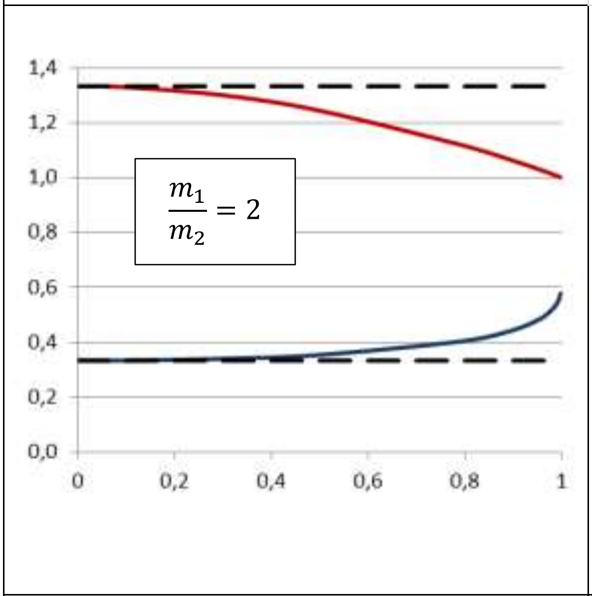
$$\delta_v = \frac{v_T(v_1, v_2)}{v_T(v_4, v_3)} - 1 \tag{6.56}$$

In Fig. 6.1 the values of the velocities v_1/c from 0.0001 to 0.999 are presented for the mass-ratio $m_1 : m_2$ of 1: 2 and 2: 1 corresponding to the starting conditions $v_2 = 0$ and $v_1 = v_2$. To ensure comparability between the examined different velocities, for any value of v_1/c the results of v_3/v_1 and v_4/v_2 were calculated and shown in a table, furthermore the findings are presented in graphical form. The graphs of the relations between the velocities show an asymptotic approach to the values of the non-relativistic cases calculated using Eq. (6.52) and Eq. (6.53), which were also inserted in the diagrams. The calculation of δ_v shows clearly, that all observers come to the same result irrespective of their velocities. This is corresponding to the examination of the non-relativistic case (see Eq. (6.52) and Eq. (6.53)).

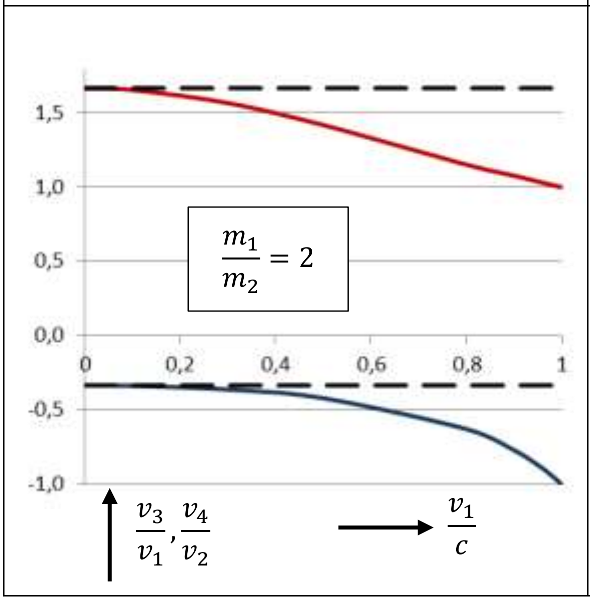
In a further examination the error range δ_v for different velocities is presented. Whereas high velocities show almost no noteworthy deviations this is changing considerably for lower values. This is caused by the decreasing accuracy during the calculation of small values because of round-off errors. Using standard spreadsheet calculation programs on a PC (such as Microsoft Excel©) the possible calculation limit is reached at values for δ_v of approximately 10^{-15}. It is not possible to calculate with higher precision, smaller values are classified as 0. The question of accuracy is also of great importance for numerical solutions; this topic is dealt with in a comprehensive way in annex D, where 3 different approaches (recursion, Newton's calculus, bisection) are described and compared.



$\frac{v_1}{c}$	$\frac{v_2}{c}$	$\frac{v_3}{v_1}$	$\frac{v_4}{v_2}$	δ_v
0,0001	-0,0001	-0,3333	0,6666	$-4{,}7\cdot10^{-8}$
0,001	0	-0,3333	0,6666	$-2{,}6\cdot10^{-10}$
0,1	0	-0,3340	0,6682	$2{,}4\cdot10^{-15}$
0,3	0	-0,3403	0,6805	$-3{,}0\cdot10^{-15}$
0,5	0	-0,3544	0,7083	0
0,8	0	-0,4054	0,8030	0
0,9	0	-0,4449	0,8679	0
0,95	0	-0,4807	0,9174	0
0,97	0	-0,5023	0,9437	0
0,99	0	-0,5392	0,9773	0
0,999	0	-0,5793	0,9973	0



$\frac{v_1}{c}$	$\frac{v_2}{c}$	$\frac{v_3}{v_1}$	$\frac{v_4}{v_2}$	δ_v
0,0001	0	0,3333	1,3333	$-2{,}6\cdot10^{-8}$
0,001	0	0,3330	1,3330	$1{,}3\cdot10^{-10}$
0,1	0	0,3340	1,3296	$-1{,}0\cdot10^{-14}$
0,3	0	0,3403	1,3050	$3{,}6\cdot10^{-15}$
0,5	0	0,3544	1,2442	$3{,}3\cdot10^{-15}$
0,8	0	0,4054	1,1159	$3{,}1\cdot10^{-15}$
0,9	0	0,4449	1,0621	0
0,95	0	0,4807	1,0317	0
0,97	0	0,5023	1,0202	0
0,99	0	0,5392	1,0070	0
0,999	0	0,5793	1,0007	$6{,}9\cdot10^{-15}$



$\frac{v_1}{c}$	$\frac{v_2}{c}$	$\frac{v_3}{v_1}$	$\frac{v_4}{v_2}$	δ_v
0,0001	-0,0001	-0,3333	1,6660	$-1{,}8\cdot10^{-8}$
0,001	-0,001	-0,3330	1,6666	$1{,}7\cdot10^{-10}$
0,1	-0,1	-0,3363	1,6550	$-5{,}6\cdot10^{-15}$
0,3	-0,3	-0,3614	1,5670	0
0,5	-0,5	-0,4194	1,4186	0
0,8	-0,8	-0,6276	1,1523	0
0,9	-0,9	-0,7697	1,0777	0
0,95	-0,95	-0,8697	1,0387	0
0,97	-0,97	-0,9174	1,0228	0
0,99	-0,99	-0,9708	1,0075	0
0,999	-0,999	-0,9970	1,0007	0

Fig. 6.1: Relativistic elastic collision for $0{,}0001 < v_1/c < 0.999$. Relations for velocities v_3/v_1 (blue), v_4/v_2 (red). Error range δ_v (For definition: see text). Non-relativistic case: dotted line.

Finally, it can be stated that during relativistic elastic collision no effects appear which would make it possible to identify the existence of a system of absolute rest in the universe. However, new attempts are made (year 2017) to identify results of this kind using precision measurements of particle mass (in this case: electrons) [45]. According to the considerations presented here it is not possible that experiments of this type can be successful at all.

6.4 Exchange of signals during and after acceleration

In this chapter it is investigated how accelerated systems behave in relativistic situations and which measurement results are obtained for other, non-accelerated observers with constant velocity. The acceleration is not generated from outside sources - e.g. by an electromagnetic field acting on a charged object - as it was investigated by H. A. Lorentz (cf. chapter 6.1), but shall be caused by thrust like it is the case for a rocket.

First, a simple situation is considered in which the system under investigation is subjected to constant acceleration, with changes in mass due to the emission of propellant gases initially being disregarded. Important results can be determined by analytical and numerical methods. Then, in a more advanced approach, consideration of the decrease in rocket mass with acceleration is added. If for the propulsion a proportional change of the ejection mass compared to the remaining rocket mass is assumed, the acceleration remains constant during a trial and the behavior is the same as in the previously investigated case.

In contrast, a constant mass decrease per time unit (as required when the classical rocket formula is used) leads to increasing acceleration values. These calculations in full scale (including acceleration and covered distance) can only be carried out numerically; a corresponding program and the results obtained with it are shown in the appendix. Further, the final velocity of a rocket, which can be calculated using the classical and relativistic rocket formula, is determined and the agreement of the results is shown.

6.4.1 Exchange of signals in systems with constant acceleration

In the following the case shall be discussed that a rocket accelerates uniformly and is observed from other inertial systems. During the acceleration process, signals are emitted by observer S inside the rocket at regular intervals of Δt_S. Further observer A also participates in the experiment and moves at the beginning of the acceleration with the same speed as S. Out of an additional inertial system, a second observer B is moving with an arbitrary velocity relative to A. Both observers A and B are recording the signals of S.

First, the acceleration of the rocket monitored by observer A is investigated. An analytical calculation is complicated by the fact that the relation for the relativistic velocity addition is not linear. During the acceleration, for the current velocity v_A the velocity change dv_A (from the point of view of A) is described by

$$v_A + dv_A = \frac{v_A + dv_S}{1 + \frac{v_A \cdot dv_S}{c^2}} \tag{6.60}$$

where dv_S represents the change of the velocity observed in the moving system S. The use of a Taylor expansion results in

$$v_A + dv_A = v_A + dv_S\left(1 - \frac{v_A^2}{c^2}\right) + \ (dv_S)^2\left(\frac{v_A^3 - v_A \cdot c^2}{c^4}\right) + + \dots \tag{6.61}$$

With a differential consideration for $dv_S \to 0$, values of $(dv_S)^2$ and higher order can be neglected. Equation (6.61) thus obtains the form

$$dv_A = dv_S\left(1 - \frac{v_A^2}{c^2}\right) \tag{6.62}$$

The applicable accelerations are now defined for both systems

$$a_S = \frac{dv_S}{dt_S} \qquad\qquad a_A = \frac{dv_A}{dt_A} \tag{6.63}$$

Furthermore

$$dt_S = \ dt_A \cdot \gamma = \frac{dt_A}{\sqrt{1 - \left(\frac{v_A}{c}\right)^2}} \tag{6.64}$$

and finally

$$a_A = \frac{dv_A}{dt_A} = \frac{dv_S}{dt_S}\left(1 - \frac{v_A^2}{c^2}\right)^{3/2} = a_S\left(1 - \frac{v_A^2}{c^2}\right)^{3/2} = \frac{a_S}{\gamma^3} \tag{6.65}$$

Thus, between a_A and a_S the same factor γ^3 appears as it was derived when determining the correlations for the occurring forces in case of relativistic mass increase (cf. chapter 6.1).

In the following, the relations between the subjectively observed times, velocities, and distances for stationary and moving observers shall be determined. For this purpose, first the velocity is considered. From eq. (6.65) follows immediately

$$dt_A = \frac{1}{a_S}\left(1 - \frac{v_A^2}{c^2}\right)^{-3/2} dv_A \tag{6.66}$$

Assuming, that values for a_S are constant and integrating Eq. (6.66), we obtain

$$t_A = \frac{v_A}{a_S}\left(1 - \frac{v_A^2}{c^2}\right)^{-1/2} + C = \frac{v_A \cdot \gamma(v_A)}{a_S} + C \tag{6.67}$$

If concrete values are used (e.g. time runs from 0 to t_A), the integration constant C equals zero. This equation describes - with subjectively constant acceleration of the rocket - the dependency between time and velocity from the point of view of A. With a given velocity, time can be determined directly, in the opposite case, a numerical procedure must be applied to determine v_A when using the equation. To avoid this, however, equation Eq. (6.67) can be extended and transformed via

$$\left(\frac{a_S \cdot t_A}{c}\right)^2 = \left(\frac{v_A \cdot \gamma(v_A)}{c}\right)^2 = \frac{\frac{v_A^2}{c^2} + 1 - 1}{1 - \frac{v_A^2}{c^2}} = \frac{1}{1 - \frac{v_A^2}{c^2}} - 1 \tag{6.68}$$

Transformed to v_A the result is

$$v_A = \frac{a_S \cdot t_A}{\sqrt{1 + \left(\frac{a_S \cdot t_A}{c}\right)^2}} \tag{6.69}$$

This representation is also found in the literature, using approaches similar to the one chosen here [32] as well as using rapidity [91]. [Note: rapidity θ describes a concept in which velocities are added up according to Galileo's principle; the relationship with relativistic velocity is $\theta = arctanh(v/c)$]. Equations (6.67) and (6.69) are equivalent and can be used depending on the computational requirements.

To calculate the time subjectively elapsing in the rocket, equations (6.64) and (6.66) are combined, yielding the relation

$$dt_S = \frac{1}{a_S}\left(1 - \frac{v_A^2}{c^2}\right)^{-1} dv_A \tag{6.70}$$

Integration leads to

$$t_S = \frac{c}{a_S}\operatorname{arctanh}\left(\frac{v_A}{c}\right) + C \tag{6.71}$$

For direct calculation of the dependency on t_A instead of v_A, Eq. (6.69) can be substituted into (6.71).

The distance travelled x_A can be calculated using Eq. (6.66) with

$$dx_A = v_A dt_A = \frac{1}{a_S}\left(1 - \frac{v_A^2}{c^2}\right)^{-3/2} dv_A \tag{6.72}$$

Integration yields

$$x_A = \frac{c^2}{a_S}\left(1 - \frac{v_A^2}{c^2}\right)^{-1/2} + C \tag{6.73}$$

In contrast to the previous cases, the integration constant must be determined here. This is done by using the boundary condition $x_A = 0$ for the velocity $v_A = 0$. Substituting in Eq. (6.73) this leads to

$$0 = \frac{c^2}{a_S}(1 - 0)^{-1/2} + C \quad \Rightarrow \quad C = -\frac{c^2}{a_S}$$

and inserted into Eq. (6.73), the final form is given by

$$x_A = \frac{c^2}{a_S}\left\{\left(1 - \frac{v_A^2}{c^2}\right)^{-1/2} - 1\right\} = \frac{c^2}{a_S}(\gamma - 1) \tag{6.74}$$

Again, the relationship between v_A and t_A from equation (6.69) can be used alternatively to obtain a direct dependence on t_A.

Equation (6.74) has the peculiarity that for small values of v_A the end results can become very inaccurate. The value of γ approaches 1 in this case; but since the value 1 is subtracted in the formula, larger errors can occur with usual calculation accuracy. It is recommended here to use a Taylor expansion where these problems do not appear. Appendix B contains a derivation in chapter B.3 and it is shown under which boundary conditions Eq. (6.74) or the Taylor method is more accurate.

Furthermore, a numerical method is also presented in this annex B, where the use of additions of relativistic velocities with sufficiently small steps leads to the same results. An analytical method is easier to use but would lead to problems in case of modifications, such as changing the acceleration during the experiment. With numerical methods, on the other hand, such a situation can be implemented easily. This becomes clear in the situation described in the next chapter, in which the real behavior of creating thrust realized by ejection of a propellant gas from a rocket and the resulting influences on the system are considered in detail.

In the following it shall be demonstrated that based on these simple correlations no contradictions will occur concerning the experimental findings of observers travelling with different velocities compared to the system, which is at rest at the start of acceleration of the rocket. The only precondition necessary is, that from the rocket signals to observers A and B are transmitted, and that these signals have a constant subjective frequency concerning the system inside the rocket. The situation of all participants is presented in the following diagram.

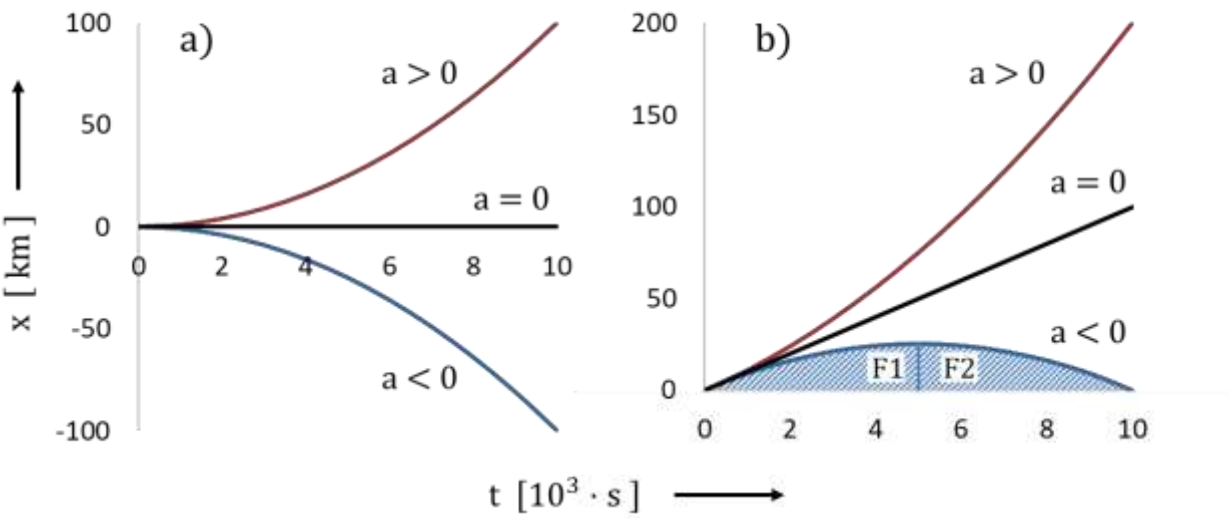


Fig. 6.2: Comparison of different acceleration conditions calculated for $a = 10\ \mathrm{m/s^2}$, $a = 0$ and $a = -10\ \mathrm{m/s^2}$
a) $v_0 = 0$, b) $v_0 = 50\ \mathrm{m/s}$

Observer B is at rest in all cases relative to the presentation of the diagram (i.e. from the point of view of A and S, he is moving relative to them at the start of the experiment with velocity v_0), while A is moving on the line $\mathrm{a} = 0$. Thus, in subplot a) with $v_0 = 0$, the results for A and B coincide, while in b) participant A is increasing the distance in relation to B with constant velocity v_0. The aim of the following calculations is to show that the values of A in part a) and also b) are identical from the point of view of B using the Lorentz equations. The principle of relativity is valid because the subjectively measured times are independent of the speed of the observers.

To prove this, Fig. 6.3 shows a situation in which subplot a) shows the rocket passing observer B (blue line in the x/t diagram), decelerates and then approaches again. In subplot b) the rocket starts from a position at rest and is accelerated uniformly. In this case, the course of an additional test participant A moving uniformly at velocity v_0 is also shown (blue line). To make the results easier to distinguish, the reference points in subplot a) have been marked with P and in b) with Q and R.

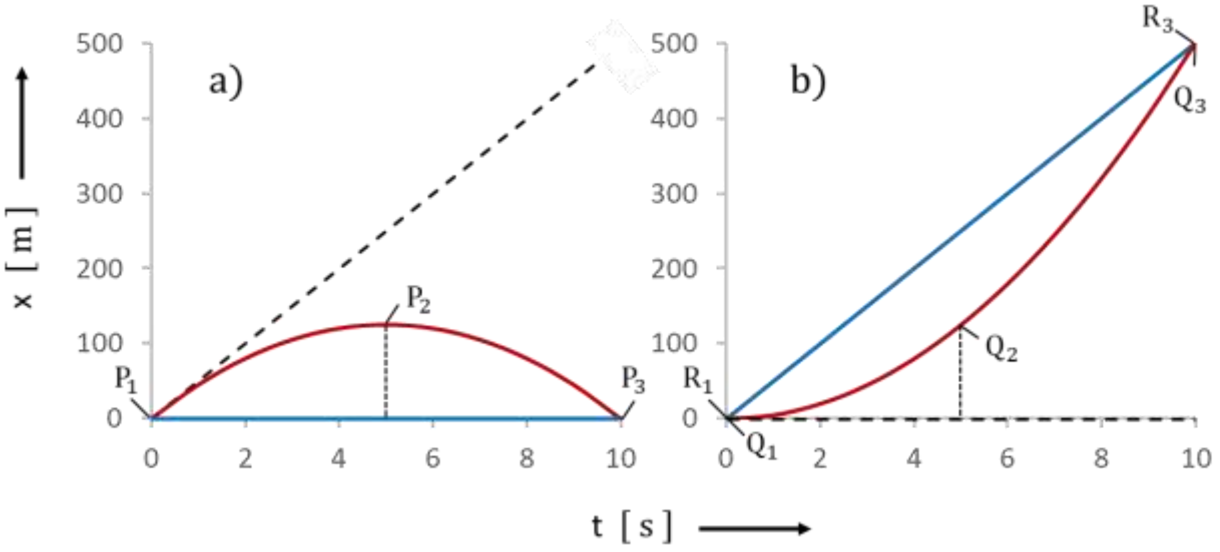


Fig. 6.3: Identical accelerations observed by different participants
a) $v_0 = 50\ \mathrm{m/s},\ a_S = -10\ \mathrm{m/s^2}$ b) $v_0 = 0,\ a_S = 10\ \mathrm{m/s^2}$

With the very small values for v_0 chosen here for the presentation in the diagram, in principle no significant deviations between relativistic and non-relativistic consideration can be provided. Therefore, calculations were carried out which are based on a system velocity of 369 km/s. As already pointed out in several other cases, this is the velocity with which our solar system is moving relative to the uniform cosmic background radiation and thus is of great interest for possible experiments to be performed. It remains to be clarified how large the difference is in the present case between relativistic and non-relativistic consideration. In order to show this, values for the non-relativistic case (Galileo) were also added to the table. As it is well known, these relations are given by

$$v = a \cdot t \tag{6.75}$$

$$x = \frac{1}{2}\, a \cdot t^2 \tag{6.76}$$

If it is assumed that a spaceship passes earth with 369 km/s and decelerates with $10\mathrm{m/s^2}$, the maximum distance would be reached at about $6{,}8 \cdot 10^6$ km (subplot a, point P_2) in non-relativistic consideration. The total time until the earth is reached again at P_3 is about 20.5 hours. The exact values and also the results calculated for a relativistic consideration are summarized in a table (Tab. 6.1).

The information included in this representation will be broken down in the following. For this purpose, it is necessary to note the sequence of the calculations. First, the subplot a) is considered:

1. $P_1 \rightarrow P_2$
   The values of $t_S(P_2)$ are calculated using Eq. (6.67), $t_A(P_2)$ is derived from Eq. (6.71) and $x_A(P_2)$ from Eq. (6.74) for the velocity $v_A = 369$ km/s. The use of Eq. (6.74) is permitted, although it was initially derived considering the case $v_A = 0$; because of symmetrical reasons first case $P_2 \rightarrow P_1$ is calculated and the result is then transferred to $P_1 \rightarrow P_2$.
2. $P_2 \rightarrow P_3$
   Because of symmetry reasons the values of $t_S(P_3)$ und $t_N(P_3)$ must be twice as large as for (P_2). The value of $x_A(P_3) = 0$ by definition.

For subplot b) the values are accordingly:

1. $Q_1 \rightarrow Q_2$
   Symmetry reasons result in $t_S(P_2) = t_S(Q_2)$, $t_A(P_2) = t_A(Q_2)$ and $x_A(P_2) = x_A(Q_2)$.
2. $Q_2 \rightarrow Q_3$
   In this case the assumption is used that subjectively within differently moved inertial systems no differences may arise at the same changes of state; this means $t_S(P_3) = t_S(Q_3)$ is set (the two fields are green and marked with arrow). If this assumption is correct, no differences may show up in a later comparison of results. First, the value for $v_A(Q_3)$ is calculated from Eq. (6.71), then $t_A(Q_3)$ from Eq. (6.67) and $x_A(Q_3)$ from Eq. (6.74).

Pos.	Kriterium	v_A	t_S	t_A	x_A
P_1		369	0	0	0
P_2	relativ.	0	36900,0186344619	36900,0279516977	6808057,73532358
P_2	Galilei	0	36900	36900	6808050
P_3	relativ.	-369	73800,0372689239	73800,0559033954	0
P_3	Galilei	-369	73800	73800	0
Q_1		0	0	0	0
Q_2	relativ.	369	36900,0186344619	36900,0279516977	6808057,73532358
Q_2	Galilei	369	36900	36900	6808050
Q_3	relativ.	737,998881935078	73800,0372689239	73800,1118068352	27232241,2567440
Q_3	Galilei	738	73800	73800	27232200
R_3	relativ.	369	73800,0559034565	73800,1118068942	27232241,2567440
R_3	Galilei		73800	73800	27232200
δ_K			$-8{,}28559 \cdot 10^{-13}$	$-8{,}001 \cdot 10^{-13}$	

Tab. 6.1: Results of calculations for $v_0 = 369$ km/s using $a_S = -10$ m/s^2 (values P) and $a_S = 10$ m/s^2 (values Q) for a non-relativistic (Galileo) and relativistic approach. Points are defined according to Fig. 6.3.

For a further evaluation, the case must be calculated, how the situation arises in subplot b) for a linearly (unaccelerated) moving observer (blue line). To realize this, the boundary condition is used that accelerated and non-accelerated observers meet at the point Q_3, i.e. the values x_N for Q_3 and R_3 must be the same in this case (these fields are also green and marked with an arrow). From

$$t_A = \frac{x_A}{v_0} \tag{6.77}$$

and

$$t_S = \frac{t_A}{\gamma} \tag{6.78}$$

the values of t_A and t_S can be calculated.

With the data determined here, a comparison between individual values can be carried out. First, the values for t_A for the accelerated and non-accelerated case are compared at point $Q_3 = R_3$, which by definition must be the same, since both start and end from the same point ($Q_1 \rightarrow Q_3$ and $R_1 \rightarrow R_3$). The values are marked in blue. Despite different calculations, they lead to approximately the same result, with the deviation according to the calculation for

$$\delta_{K1} = \frac{t_A(Q_3)}{t_A(R_3)} - 1 \tag{6.79}$$

to be determined. The same behavior occurs when the values for $t_A(P_3)$ and $t_S(Q_3(L)$ are compared (marked in yellow)

$$\delta_{K2} = \frac{t_A(P_3)}{t_S(R_3)} - 1 \tag{6.80}$$

These must be equal for the following reason: The stationary observer in subplot a) determines that the passing rocket arrives at his position again after uniform negative acceleration at the time t_A. The uniformly moving observer in subplot b) must subjectively observe the same behavior. For the situation of an observer at rest in subplot b), represented by the course of the dashed line, the value for t_A is higher in this case, but can be traced back to the subjective measured value of the moving system by simple division by γ. No relevant calculation differences can be determined here.

With the boundary conditions selected here using $v_A = 369$ km/s, deviations of approx. $8 \cdot 10^{-13}$ occur for δ_K. If, on the other hand, higher values for v_A are selected, as e.g. in Tab. 6.2 with $v_A = 0{,}5c$, no deviations are detectable within the scope of the calculation accuracy, but with smaller values for v_A they increase. This is due to the occurrence of very small values of γ, especially in Eq. (6.74). At small velocities, the value for γ is only slightly larger than 1; if the value of 1 is subtracted from this, large deviations can result depending on the accuracy of the calculation. This effect is shown in more detail in annex B, chapter B.3 and for this purpose a significant improvement of the accuracy is demonstrated by using a Taylor expansion.

Instead of the analytical approach chosen here, the regularities can also be determined numerically. A procedure for this is compiled in Annex B. If the occurring deviations are considered, an advantage for the numerical procedure is shown with low values of v_A, with higher velocities it is the other way round; the accuracy depends beyond that substantially on the number of the selected iteration steps. After performing the numerical calculations, it is shown here that the subjectively existing acceleration between motionless and moving observer differs by a factor γ^3; in contrast to the analytical method, where this was determined by basic considerations, this is a result of the calculations performed. In the Annex B the results are presented in detail. Also added is a comparison with results of the numerical method from Annex C, in which the amount of propellant gas ejected was kept constant in relation to the residual mass of the rocket, thus achieving uniform acceleration.

Pos.	Kriterium	v_A	t_S	t_A	x_A
P_1		149896,229	0	0	0
P_2	relativ.	0	16467783,9204409	17308525,6327320	1390379100217,26
P_2	Galilei	0	14989622,9	14989622,9	6808050
P_3	relativ.	-149896,229	32935567,8408818	34617051,2654639	0
P_3	Galilei	-149896,229	29979245,8	29979245,8	0
Q_1		0	0	0	0
Q_2	relativ.	149896,229	16467783,9204409	17308525,6327320	1390379100217,26
Q_2	Galilei	149896,229	14989622,9	14989622,9	6808050
Q_3	relativ.	239833,96640	32935567,8408818	39972327,7333333	5991701191578,78
Q_3	Galilei	299792,458	29979245,8	29979245,8	4493775893684,09
R_3	relativ.	149896,229	34617051,2654639	39972327,7333333	5991701191578,78
R_3	Galilei	299792,458	29979245,8	29979245,8	4493775893684,09
δ_K			0	0	

Tab. 6.2: Results of calculations for $v_0 = 0{,}5c$ using $a_S = -10\ \mathrm{m/s^2}$ (values P) and $10\ \mathrm{m/s^2}$ (values Q) for a non-relativistic (Galileo) and relativistic approach. Points are defined according to Fig. 6.3.

An evaluation of the chosen general conditions reveals at first sight that a rocket technology generating the required thrust long enough is not existing today; with such a system it would be possible to reach Mars in a few days. This becomes even clearer if a long journey is considered under the conditions chosen here. If it is assumed that a body of 100 tons with constant acceleration of 1g crosses the galaxy (100,000 light years, subjective time on board: approx. 12 years), the rocket with a propellant density of 70 kg/m^3 would have to have a size of 14 x 14 x 14 km^3 at departure, even if an optimal conversion of mass into kinetic energy is assumed [91]. This does not include any statements on the deceleration of the rocket after the journey or on the influence of micrometeorites and gas causing a speed reduction, or the protection of the passengers by additionally required masses due to necessary shielding devices.

Despite the obvious impossibility of implementation on an industrial scale, however, the results calculated here are unambiguous and show that - although the influence is small - they must be taken into account when even small acceleration phases are considered.

Finally, questions of the influence of acceleration on the measurements shall be examined in general. According to the Theory of General Relativity it is not possible for an observer to decide with measurements in a closed system, whether he is exposed to an acceleration effect caused by increasing velocity or by a gravitation field. Although it is not without controversy that additional (gravitational) time dilatation will appear in accelerated systems, the potential effect shall be estimated to complete a general consideration.

For the conditions chosen here with an acceleration value of 10 m/s^2, which corresponds approximately to the effect of the earth's acceleration due to gravity of 9.81 m/s^2, a time dilation of about $7 \cdot 10^{-10}$ results, which has been confirmed by many measurements [80]. If this value is multiplied by the total time from Tab. 6.2, an effect of $5{,}17 \cdot 10^{-5}$ s results. This would mean that the calculated time difference between relativistic and non-relativistic consideration is extended by a value of 0.28%. Thus, because of the small deviation, this potential effect can be neglected here.

6.4.2 Relativistic rocket propulsion

Now the question arises, how a rocket behaves in reality, which is accelerated by outflowing gas and accordingly loses mass. An observer B, who monitors this process from another inertial frame and measures the velocity v_0 for S and A at the beginning of the experiment, will find differences to the measurements of S due to the time dilation and the relativistic mass increase, namely

1. The quantity of the gas-molecules generating the repulsion force is reduced by the factor $\gamma(v_0)$ per time unit.
2. The mass of any single molecule of the gas is increased by the factor $\gamma(v_0)$.
3. The remaining mass of the rocket is increased by the factor $\gamma(v_0)$.
4. The speed of the outflowing gas corresponds to the theorem of relativistic addition of velocities.
5. The elapsing time between outgoing signals is increased by the factor $\gamma(v_0)$.
6. The total time for acceleration during an experiment is increased by the factor $\gamma(v_0)$.

For the exact determination of the situation, all influences related to these criteria must be calculated with respect to the reduction of the rocket mass due to the gas ejection for propulsion. These conditions are considered for cases with constant gas ejection (which leads to a steady increase in acceleration) and with constantly reduced gas ejection (to ensure constant acceleration).

The relativistic momentum is used to establish the equations relevant to solve this problem. It is determined in general that all functions referring to the outflowing gas are marked with f'; relations connected with the moving rocket, on the other hand, are represented without this marking.

Following this general definition, the relativistic momentum of a rocket before starting acceleration is

$$p_0 = m_0 v_0 \gamma_0 \tag{6.81}$$

where v_0 is the velocity of the rocket relative to a reference frame at the start of the trial. After the first step the relation changes to

$$p_1 = m_1 v_1 \gamma_1 \tag{6.82}$$

and the values for step 1 are calculated as follows:

1. It is assumed that during the first step of acceleration the rocket is losing mass Δm_0 with the jet velocity v'_0; the gas used to form the high-speed jet to generate the repulsion force is generally called "propellant mass".
2. The momentum of the rocket p_1 (related to the remaining mass $m_1 = m_0 - \Delta m_0$) and p'_1 of the propellant mass Δm_0 are added and set equal to the momentum p_0 of the rocket (using of the law of conservation of momentum). From this, the changing velocity of the rocket is calculated. This results in

$$p_1 + p'_1 = (m_0 - \Delta m_0) v_1 \gamma_1 + \Delta m_0 v'_1 \gamma'_1 = m_0 v_0 \gamma_0 \tag{6.83}$$

and generally

$$p_K + p'_K = (m_{K-1} - \Delta m_{K-1}) v_K \gamma_k + \Delta m_{K-1} v'_K \gamma'_K = m_{K-1} v_{K-1} \gamma_{K-1} \tag{6.84}$$

The values for v and v' show in different directions (this is explaining the "+" in the formula). Relative to the rocket, the gas flow maintains at a constant speed of v'_0. The relativistic addition of velocities is leading to

$$v'_{K+1} = \frac{v_{K+1} + v'_0}{1 + \frac{v_{K+1} v'_0}{c^2}} \tag{6.85}$$

Using the equations (6.84) and (6.85) for every step K the velocity of the rocket can be calculated; this means the complete numerical evaluation is following a nested loop with a subroutine for any v_K.

To perform such a calculation, programming was done in Visual Basic (VBA). The VBA program code is compiled in Annex C with the corresponding formulas and a flow chart. The main purpose of these calculations is the comparison of systems which are at rest at the time of the start of the trial to those which are relatively moved. For this purpose, two exemplary calculation variants were programmed, whereby firstly the acceleration and in the second case the outflow velocity of the propellant mass were kept constant. The differences associated with both concepts are presented in the following.

a) Propellant mass proportional to the remaining mass of the rocket

The precondition of propellant mass proportional to the remaining mass of the rocket results in constant acceleration values for the rocket over the entire observation period. This situation corresponds to the case already described in chapter 6.4.1.

Table 6.3 shows the results of two calculations with $v_0 = 0$ and $v_0 = 0{,}5c$ as initial velocities. The selected values are quite different and this also the case for the results. In order to enable a comparison of the values with each other, the final velocity of the rocket from the view of an observer at rest was defined as the difference $v_T = v_N - v_0$. The value t_T is the total time, which results subjectively from the view of the unmoved system when applying the Lorentz equations for an observer moving with system velocity v_0 until the arrival of a signal from the rocket.

In addition, the distance x_N covered by the rocket from the view of the stationary observer up to the emission of the impulse is listed. Furthermore, the result for the remaining mass m_N of the rocket after completion of the experiment is shown (related to the initial value $m_0 = 1$). In addition, the values for the accelerations a_N and also the calculations for $\gamma^3 a_N$ are presented.

K	N	v_T	t_T	m_N	x_N	a_N	$\gamma^3 a_N$
1	10	3,9999999999413	400,00266852469	0,3486784401000	800,0000000579	9,9999999975878	10,000000000258
2	10^2	3,9999999999407	400,00266852463	0,3660323412732	800,0000000592	9,9999999973564	10,000000000027
3	10^3	3,9999999999408	400,00266852463	0,3676954247710	800,0000000594	9,9999999973343	10,000000000005
4	10^4	3,9999999999424	400,00266852463	0,3678610464329	800,0000000596	9,9999999973292	10,000000000000
5	10^5	3,9999999999581	400,00266852464	0,3678776017666	800,0000000616	9,9999999975070	10,00000000177
6	10^6	4,0000000001169	400,00266852155	0,3678792572316	800,0000000717	9,9999999975070	10,00000000177
7	10^7	4,0000000016930	400,00266859493	0,3678794227775	800,0000004935	10,000000005125	10,000000007795

	δv_T	δt_T	δm_N	δx_N	δa_N	$\delta \gamma^3 a_N$
1/2	$-5,7021 \cdot 10^{-13}$	$-5,7980 \cdot 10^{-11}$	$1,7354 \cdot 10^{-2}$	$1,2930 \cdot 10^{-9}$	$-2,3141 \cdot 10^{-10}$	$-2,3141 \cdot 10^{-10}$
2/3	$1,3989 \cdot 10^{-13}$	$-1,0232 \cdot 10^{-12}$	$1,6631 \cdot 10^{-3}$	$1,3006 \cdot 10^{-10}$	$-2,2089 \cdot 10^{-11}$	$-2,2089 \cdot 10^{-11}$
3/4	$1,5601 \cdot 10^{-12}$	0	$1,6562 \cdot 10^{-4}$	$2,1396 \cdot 10^{-10}$	$-5,0804 \cdot 10^{-12}$	$-5,0804 \cdot 10^{-12}$
4/5	$1,5710 \cdot 10^{-11}$	$8,98 \cdot 10^{-12}$	$1,6555 \cdot 10^{-5}$	$2,0430 \cdot 10^{-9}$	$1,7776 \cdot 10^{-10}$	$1,7776 \cdot 10^{-10}$
5/6	$1,5883 \cdot 10^{-10}$	$-3,09 \cdot 10^{-9}$	$1,6555 \cdot 10^{-6}$	$1,0087 \cdot 10^{-8}$	0	0
6/7	$1,5760 \cdot 10^{-9}$	$7,3485 \cdot 10^{-8}$	$1,6555 \cdot 10^{-7}$	$4,2175 \cdot 10^{-7}$	$7,6180 \cdot 10^{-9}$	$7,6180 \cdot 10^{-9}$

K	N	v_T	t_T	m_N	x_N	a_N	$\gamma^3 a_N$
1	10	2,9999709803087	400,00266851663	0,3486784401000	69235026,29063	6,4951038669616	10,000066715204
2	10^2	2,9999700801272	400,00266851582	0,3660323412732	69235026,29036	6,4950395173348	9,9999676404299
3	10^3	2,9999699985783	400,00266851575	0,3676954247710	69235026,29034	6,4950389552623	9,9999667750399
4	10^4	2,9999700695917	400,00266851581	0,3678610464329	69235026,29036	6,4950385228968	9,9999661093612
5	10^5	2,9999708701507	400,00266851649	0,3678776017666	69235026,29059	6,4950356404560	9,9999616715177
6	10^6	2,9999825792620	400,00266852517	0,3678792572316	69235026,29360	6,4949779917004	9,9998729144571
7	10^7	3,0001320510055	400,00266865907	0,3678794227775	69235026,33995	6,4955544754057	10,000760496920

	δv_T	δt_T	δm_N	δx_N	δa_N	$\delta \gamma^3 a_N$
1/2	$-9,0018 \cdot 10^{-7}$	$-8,1502 \cdot 10^{-10}$	$1,7354 \cdot 10^{-2}$	$-2,6439 \cdot 10^{-4}$	$-6,4350 \cdot 10^{-5}$	$-9,9075 \cdot 10^{-5}$
2/3	$-8,1549 \cdot 10^{-8}$	$-7,2987 \cdot 10^{-11}$	$1,6631 \cdot 10^{-3}$	$-2,5108 \cdot 10^{-5}$	$-5,6207 \cdot 10^{-7}$	$-8,6539 \cdot 10^{-7}$
3/4	$7,1013 \cdot 10^{-8}$	$6,4006 \cdot 10^{-11}$	$1,6562 \cdot 10^{-4}$	$2,2203 \cdot 10^{-5}$	$-4,3237 \cdot 10^{-7}$	$-6,6568 \cdot 10^{-7}$
4/5	$8,0056 \cdot 10^{-7}$	$6,7701 \cdot 10^{-10}$	$1,6556 \cdot 10^{-5}$	$2,3431 \cdot 10^{-4}$	$-2,8824 \cdot 10^{-6}$	$-4,4378 \cdot 10^{-6}$
5/6	$1,1709 \cdot 10^{-5}$	$8,6780 \cdot 10^{-9}$	$1,6555 \cdot 10^{-6}$	$3,0045 \cdot 10^{-3}$	$-5,7649 \cdot 10^{-5}$	$-8,8757 \cdot 10^{-5}$
6/7	$1,4947 \cdot 10^{-4}$	$1,3391 \cdot 10^{-7}$	$1,6555 \cdot 10^{-7}$	$4,6355 \cdot 10^{-2}$	$5,7648 \cdot 10^{-4}$	$8,8758 \cdot 10^{-4}$

Tab. 6.3: Values of v_T, t_T, m_N, x_N, a_N, $\gamma^3 a_N$ for proportional reduction of propellant mass.
Top: $v_0 = 0$, bottom: $v_0 = 0{,}5\, c$ (149.896,458 km/s).
$\Delta m_0 = 0{,}25\%/\mathrm{s}$, $t_S = 400\mathrm{s}$. The values for m_N are normalized to 1.
Values for v_T in km/s, t_T in s, x_N in km, a_N and $\gamma^3 a_N$ in m/s^2.

For the calculations a loss of propellant mass per time unit of $\Delta m_0 = 0{,}25\%/\mathrm{s}$ was specified. This leads to an acceleration of $10\mathrm{m/s^2}$ and thus a comparability with the other already performed calculations is given. The experimental time chosen was $t_S = 400\mathrm{s}$, and this leaves the realistic magnitude of a residual mass of almost 37% of the initial value after the completion of the experiment. For better evaluation, the deviations between the values $\delta v_T = v_T(K)$ and $v_T(K-1)$ are shown according to the relationships also used elsewhere (e.g., as defined in Eq. (6.79)), and in the same way for δt_T, δm_N, δx_N, a_N and $\gamma^3 a_N$, where K corresponds here in each case to a potency of ten in the number of calculation steps between 10 and 10^7 (cf. Tab. 6.3). First, it should be noted in principle that the values for δv_T, δt_T and δx_N show unsystematic fluctuations and exhibit the smallest deviations from each other considering the number of iteration steps between $\mathrm{N} = 10^2$ and 10^4. Hereby it is clear that the visible differences are not caused by a physically explainable effect, but only by the use of the numerical method.

Furthermore, it can be seen that the value of the remaining mass m_N becomes more accurate with each increase by a factor of 10 in the number of iteration steps (Iteration $10^3 \rightarrow 10^4 = 1.6562 \cdot 10^{-4}$; $10^4 \rightarrow 10^5 = 1.6566 \cdot 10^{-5}$ and so on, see Tab. 6.3). This is not of further importance here and therefore an evaluation is not carried out at this point; however, this changes in the following considerations for the case of constant propellant mass and will be further investigated there.

The results of the calculations for $\gamma^3 a_N$ show again that the ratio for the accelerations between differently moving observers reveals the factor γ^3.

The determination made here with a proportional loss of propellant mass with respect to the residual mass of the rocket allows a direct comparison with the analytical and numerical results from Section 6.4.1. and the conformity proves to be very good. A detailed evaluation is presented in Annex B.4.

b) Propellant mass constant

This case proves to be significantly more complex with regard to the evaluation compared to the situation discussed before. This is due to the fact that the values of v_T, t_T and x_N, which are important for the observation, show the same behavior as m_N before and become more precise with increasing number of iteration steps. Therefore, they must be analyzed in particular (in contrast to the case before, m_N does not show this behavior here!).

This becomes clear when considering the case shown in Tab. 6.4. In the upper part of the table, as before, the results of the calculations of the relevant values are given, below – marked with section I – the compilation of the deviations δv_T, δt_T, δm_N und δx_N follows. The first and the last calculation deviate in values from the systematics of the other results and were not considered further. Therefore, only the blue colored fields were used for final calculations and the values reproduced in section II were extrapolated from them. The results presented in the lower part of the table show the outcome of these calculations. The mass reduction was set to $\Delta m_0 = 0{,}5\%/\mathrm{s}$, which leads to a test duration of $t_0 = 100\mathrm{s}$ for the final mass value of 50% desired here.

In the Annex C, besides the derivation of the program structure, further results of the calculations for different boundary conditions were presented in the tables C.2, C.3 and C.4.

In addition to the figures for the system velocity of $v_0 = 0$ discussed here, calculated values for 369 km/s plus 2,000 km/s and 10,000 km/s were also added to provide a better overview. In these cases, a lower remaining mass after the test was also determined with a rest of 10%.

N	v_T	t_T	m_N	x_N	Δt_o
10	2,67508561278727	100,000397329364	0,500000000000000	119,116010675216	10
10^2	2,76261372200990	100,000408141269	0,500000000000000	122,357320955608	1
10^3	2,77158897232187	100,000409292747	0,500000000000055	122,702523336750	10^{-1}
10^4	2,77248872482278	100,000409408634	0,500000000000055	122,737265091767	10^{-2}
10^5	2,77257872237194	100,000409420246	0,499999999996724	122,740741494222	10^{-3}
10^6	2,77258772224753	100,000409422400	0,500000000041133	122,741089155569	10^{-4}
10^7	2,77258862465211	100,000409440862	0,499999999708066	122,741124020357	10^{-5}
	δv_T	δt_T	δm_N	δx_N	
$\bar{x}$	8,9931	$1,4064 \cdot 10^{-3}$		$3,4698 \cdot 10^2$	
10^2	$8,7528 \cdot 10^{-2}$	$1,0812 \cdot 10^{-5}$	0	3,2413	I
10^3	$8,9753 \cdot 10^{-3}$	$1,1515 \cdot 10^{-6}$	$5,4956 \cdot 10^{-14}$	$3,4520 \cdot 10^{-1}$	
10^4	$8,9975 \cdot 10^{-4}$	$1,1589 \cdot 10^{-7}$	0	$3,4742 \cdot 10^{-2}$	
10^5	$8,9998 \cdot 10^{-5}$	$1,1612 \cdot 10^{-8}$	$-3,3309 \cdot 10^{-12}$	$3,4764 \cdot 10^{-3}$	
10^6	$8,9998 \cdot 10^{-6}$	$2,1540 \cdot 10^{-9}$	$4,4409 \cdot 10^{-11}$	$3,4766 \cdot 10^{-4}$	
10^7	$9,0240 \cdot 10^{-7}$	$1,8462 \cdot 10^{-8}$	$-3,3307 \cdot 10^{-10}$	$3,4865 \cdot 10^{-5}$	
10^8	$8,9931 \cdot 10^{-8}$	$1,4069 \cdot 10^{-11}$		$3,4698 \cdot 10^{-6}$	
10^9	$8,9931 \cdot 10^{-9}$	$1,4069 \cdot 10^{-12}$		$3,4698 \cdot 10^{-7}$	
10^{10}	$8,9931 \cdot 10^{-10}$	$1,4211 \cdot 10^{-13}$		$3,4698 \cdot 10^{-8}$	
10^{11}	$8,9931 \cdot 10^{-11}$	0		$3,4698 \cdot 10^{-9}$	
10^{12}	$8,9933 \cdot 10^{-12}$	0		$3,4699 \cdot 10^{-10}$	
10^{13}	$8,9928 \cdot 10^{-13}$	0		$3,4703 \cdot 10^{-11}$	
10^{14}	$9,0150 \cdot 10^{-14}$	0		$3,4674 \cdot 10^{-12}$	II
10^{15}	$8,8818 \cdot 10^{-15}$	0		$3,4106 \cdot 10^{-13}$	
10^{16}	0	0		0	
	v_T	t_T		x_N	
10^7	2,77258862155768	100,000409422541		122,741123853607	
10^8	2,77258871148869	100,000409422555		122,741127323411	
10^9	2,77258872048179	100,000409422556		122,741127670391	
10^{10}	2,77258872138110	100,000409422556		122,741127705089	
10^{11}	2,77258872147103	100,000409422556		122,741127708559	
10^{12}	2,77258872148003	100,000409422556		122,741127708906	
10^{13}	2,77258872148093	100,000409422556		122,741127708941	
10^{14}	2,77258872148102	100,000409422556		122,741127708944	
10^{15}	2,77258872148102	100,000409422556		122,741127708945	
10^{16}	2,77258872148102	100,000409422556		122,741127708945	

Tab. 6.4: Values of v_T, t_T, m_N and x_N for linear reduction of propellant mass. Section I: Iterations, Section II: Extrapolated. All values in km and s. Calc.-Type: “A1”, $v_0' = -4$ km/s, $\Delta m_0 = 0{,}5\%/s$, $t_0 = 100$s, $v_0 = 0$

Again, the most important statement results from the comparison of the calculated values for t_T, which represent the signal propagation times until reaching an observer moving with v_0, calculated in view of the system at rest. For a better comparison of the times, here as in other cases, the comparative formula

$$\delta t_T = \frac{t_T(v_K)}{t_T(v_{K-1})} - 1 \tag{6.86}$$

was chosen. Table 6.5 shows the results of values for t_T and δt_T, where the calculation was based on t_T using iteration steps of $\mathrm{N} = 10^{16}$. No systematic deviations can be found when results for different system velocities are compared.

	1		2		3	
v_0	t_T	δt_T	t_T	δt_T	t_T	δt_T
0	100,000409422556		1.000,00992905474		10.002,4827416511	
369	100,000409421505	$1{,}05 \cdot 10^{-11}$	1.000,00992904501	$9{,}73 \cdot 10^{-12}$	10.002,4827411418	$5{,}09 \cdot 10^{-11}$
2000	100,000409421509	$-3{,}40 \cdot 10^{-14}$	1.000,00992904483	$1{,}76 \cdot 10^{-13}$	10.002,4827388902	$2{,}25 \cdot 10^{-10}$
10000	100,000409421471	$3{,}74 \cdot 10^{-13}$	1.000,00992904385	$9{,}82 \cdot 10^{-13}$	10.002,4827278462	$1{,}10 \cdot 10^{-9}$

Tab. 6.5: t_T and δt_T with constant propellant mass per time unit for different v_0.
Δm_0 is normalized to 1.
1: $v_0' = -4\ \mathrm{km/s}$, $\Delta m_0 = 0{,}5\%/s$, $t_0 = 100s$
2: $v_0' = -4\ \mathrm{km/s}$, $\Delta m_0 = 0{,}09\%/s$, $t_0 = 1.000s$
3: $v_0' = -100\ \mathrm{km/s}$, $\Delta m_0 = 0{,}009\%/s$, $t_0 = 10.000s$

For the consideration of the final velocity v_T the possibility of a comparison with the values determined according to the classical rocket formula arises. The formula derived by K. E. Tsiolkovsky in 1903 is based on the non-relativistic momentum equation and aims to calculate the terminal velocity of a rocket as a function of the exit velocity of the gas for a constant propellant mass. For non-relativistic consideration with $v \ll c$, first Eq. (6.85) is reduced to

$$v_K' = v_K + v_0' \tag{6.87}$$

To solve the equation Eq. (6.84), the stipulation that $\gamma = 1$ (not relativistic) applies. Since the mass of the rocket decreases with increasing index K, but the velocity rises, the following relations apply additionally

$$m_K = m_{K-1} - \Delta m_{K-1} \qquad\qquad v_K = v_{K-1} + \Delta v_{K-1}$$

In addition, for differential consideration the following definitions are introduced:

$$m_K \to m \qquad\qquad \Delta m \to \mathrm{d}m$$

$$v_K \to v \qquad\qquad \Delta v \to \mathrm{d}v$$

This results in the following approach for Eq. (6.84):

$$(m + \mathrm{d}m - \mathrm{d}m)v + \mathrm{d}m(v + v_0') = (m + \mathrm{d}m)(v - \mathrm{d}v) \tag{6.88}$$

$$mv + v\mathrm{d}m + v_0'\mathrm{d}m = mv - m\mathrm{d}v + v\mathrm{d}m - \mathrm{d}m\mathrm{d}v \tag{6.89}$$

and because of $\mathrm{d}m\mathrm{d}v \to 0$

$$m\mathrm{d}v + v_0'\mathrm{d}m = 0 \tag{6.90}$$

If mass and velocity of the outflowing gas (and thus the momentum) are kept constant, the integration of eq. (6.90) leads to the classical rocket formula

$$\int_0^v \mathrm{d}v = -\,v_0' \int_{m_0}^{m} \frac{\mathrm{d}m}{m} \tag{6.91}$$

$$v = v_0'\, ln\left(\frac{m_0}{m}\right) \tag{6.92}$$

where m_0 is the mass at the start from an unmoved platform. If the starting point is moving, the velocities are simply added. This becomes necessary e.g. at the drop of a rocket stage, when the mass decreases and also the momentum changes.

Besides the classical rocket formula according to Tsiolkovsky, also a relativistic rocket formula exists. This was derived in 1946 by J. Akeret [90]. The derivation is clearly more complex and requires additionally the use of the energy conservation theorem; the derivation is shown in the appendix C under point C.4. The result of this relativistic rocket equation according to Eq. (C.33) is

$$\frac{v}{c} = \frac{1 - \left(\frac{m}{m_0}\right)^{2v_0'/c}}{1 + \left(\frac{m}{m_0}\right)^{2v_0'/c}} \tag{6.93}$$

If the classical and/or the relativistic rocket equations v_R are taken as a limiting case to the presented solution of the numerically derived relativistic rocket formulas, and the results from the values for v_T calculated in appendix C, tables C2, C3 and C4 are related to each of them, the following values for a comparison can be obtained

$$\delta_R = \frac{v_R}{v_T} - 1 \tag{6.93}$$

The results of these calculations are shown in Fig. 6.4. First, it becomes clear that for low system velocities, especially in the case $v_0 = 0$, no sufficient accuracy is achieved for iteration steps from $\mathrm{N} = 10$ to $\mathrm{N} = 10^7$ and they are therefore to be considered only with restrictions. On the other hand, if the extrapolated values calculated up to $\mathrm{N} = 10^{16}$ are added, a significantly improved result is obtained. When the values for classical and relativistic rocket formulas are compared, no differences can be found for $v_0' = -\,4$ km/s, while for $v_0' = -100$ km/s, discrepancies can be seen for small system velocities ($v_0 = 0$ und 369 km/s). To show the differences, the results for the classical rocket formula (Tsiolkovsky) and relativistic (Akeret) were presented separately in subplots c) and d).

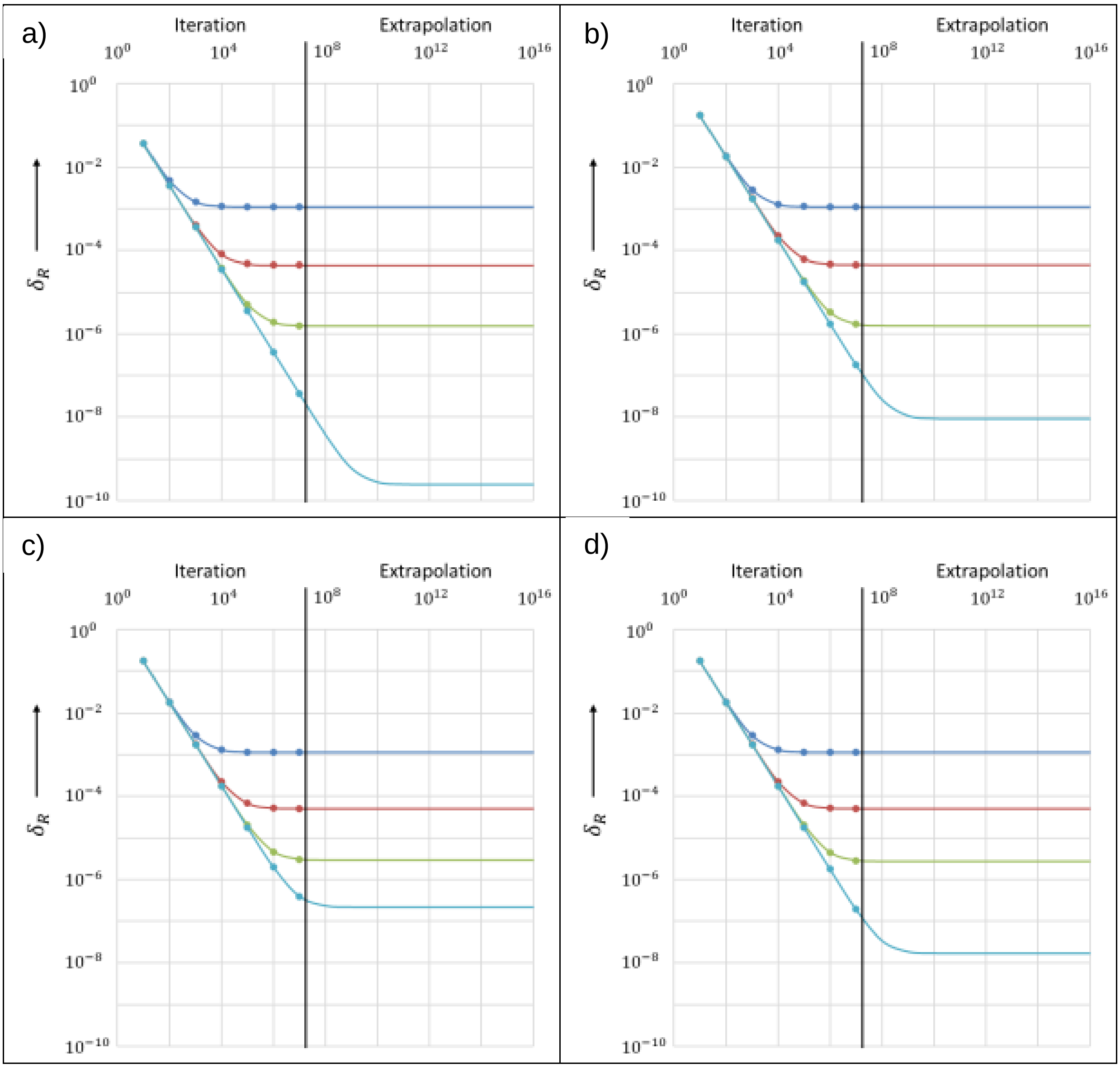


Fig. 6.4: Dependency δ_R between relativistic und classical rocket formula related to the number of iteration steps acc. to Tab. C.2, C3 and C4.
a) $v_0' = -\,4\ \mathrm{km/s},\ \ \Delta m_0 = 0{,}5\%/\mathrm{s},\ \ t_0 = 100\mathrm{s}$
b) $v_0' = -\,4\ \mathrm{km/s},\ \ \Delta m_0 = 0{,}09\%/\mathrm{s},\ \ t_0 = 1.000\mathrm{s}$
c) and d) $v_0' = -100\ \mathrm{km/s},\ \ \Delta m_0 = 0{,}009\%/\mathrm{s},\ \ t_0 = 10.000\mathrm{s}$
c) classic (acc. to K. E. Tsiolkowski), d) relativistic (acc. to J. Akeret).
a) to d) at the bottom $v_0 = 0$ then ascending $v_0 = 369,\ \ 2000, 10000\ \mathrm{km/s}$
Δm_0 normalized to 1.

To evaluate the behavior at higher velocities, results from the numerical rocket equations are compared with corresponding values from the classical and relativistic rocket formulas. In Tab. 6.6, the calculated values of the final velocity are entered for the parameters v_0'/c (gas velocity of a rocket in relation to the speed of light) and for the ratio of the masses at the final stage compared to the start.

An evaluation shows that up to a velocity of the propellant gas of 0.01c, there are no major differences between the calculations. At 0.1c the differences between the classical rocket formula and the other two variants already become clear and at 0.5c the speed of light is exceeded according to the classical nonrelativistic method at a mass release of approx. 90%. The values according to J. Akeret and those of the own numerical calculation, which of course remain below the speed of light, hardly differ.

v_0'/c	A	B	C	A	B	C	A	B	C
m/m_0	0,01	0,01	0,01	0,1	0,1	0,1	0,5	0,5	0,5
0,5	0,006931	0,006931	0,006932	0,069315	0,069204	0,069432	0,346574	0,333333	0,362675
0,2	0,016094	0,016093	0,016093	0,160944	0,159568	0,159727	0,804719	0,666667	0,681958
0,1	0,023026	0,023022	0,023022	0,230259	0,226274	0,226122	1,151293	0,818182	0,818378
0,01	0,046052	0,046019	0,046021	0,460517	0,430506	0,428238	2,302585	0,980198	0,973447
0,001	0,069078	0,068968	0,069009	0,690776	0,598480	0,593888	3,453878	0,998002	0,996217

Tab. 6.6: End velocity of a rocket (values in relation to the speed of light) depending on the calculation method
Parameter top: Values for propellant gas (values in relation to the speed of light)
Parameter left: Ratio of final mass to the mass at the start
A: Classical, acc. to K. E. Tsiolkowski
B: Relativistic, acc. to J. Akeret
C. Numerical, calculation acc. to annex C ($\Delta m_0 = 10^{-5}$ %/s, $\Delta t_S = 100$s)

The essential difference between analytical and numerical calculation is that for the analytical method no output quantity of the gas *per time unit* must be given and that therefore the result is independent of the acceleration occurring during a rocket launch. Therefore, there is also no information about which distance the rocket has covered in which time. This means, only the data determined according to the described numerical method can be used for the previously performed calculations; the analytical rocket formula does not provide the necessary information.

To illustrate this, results for gas ejection velocities of $v_0' = -0{,}5$c and $v_0' = -100$ km/s are presented below. In Tab. 6.7, gas ejection rates of $\Delta m_0 = 10^{-7}$ to 10^{-4}/s (corresponding to 10^{-5} and 10^{-2} %/s) were selected for the numerical determination and the values of v_T, t_T, x_K and a_K were calculated on these. First, it should be noted that in all cases the final velocity v_T remains constant for the respective gas exit velocity. When the gas ejection rate (per time unit) is increased by a factor of ten, the values for the total duration of the experiment t_T as well as the distance traveled x_K increase by the same factor. The acceleration a_K, on the other hand, decreases by the same amount.

Finally, an essential difference between the numerical method and the relativistic rocket formula must be pointed out. While the latter was derived using the law of conservation of energy, the numerical method (as well as the classical rocket formula according to Tsiolkovsky) is based exclusively on the law of conservation of momentum. For the calculation, this means that the momentum of the propulsion gas could in theory be increased unlimited by approaching the speed of light more and more, and thus extremely high rocket velocities could be achieved connected with a low mass output. However, in reality this is not possible, because for the acceleration of the propellant gas considerable amounts of energy (and thus because of $E = mc^2$ additional mass losses) would be needed, which are not considered in the calculation. For these extreme values, therefore, the numerical method presented cannot be used.

	$v_0' = -0{,}1c$				$v_0' = -100 km/s$			
Δm_0	v_T	t_T	x_K	a_K	v_T	t_T	x_K	a_K
10^{-7}	67789,6421	9713871,60	2017951968	27,5336472	230,263962	900233,452	669757541	0,0999999043
10^{-6}	67789,6421	971387,160	2017951968	275,336472	230,263962	900233,452	66975754,1	0,9999990426
10^{-5}	67789,6421	97138,7160	2017951968	2753,36472	230,263962	90023,3452	6697575,41	9,9999904265
10^{-4}	67789,6421	9713,87160	201795196,8	27533,6472	230,263962	9002,33452	669757,541	99,999904265

Tab. 6.7: End velocity v_T, total time t_T, covered distance x_K and acceleration a_K as a function of the gas ejection velocity and the gas quantity Δm_0 (per time unit). v_T in $\mathrm{km/s}$, t_T in s, x_K in km, a_K in $\mathrm{m/s^2}$, Δm_0 in $1/s$ (normalized to 1)

The problem of determining the energy requirement for rocket propulsion systems has been discussed for a long time and can be solved by defining various loss factors. As an example, the representation used by U. Walter [91] is given in Fig. 6.5.

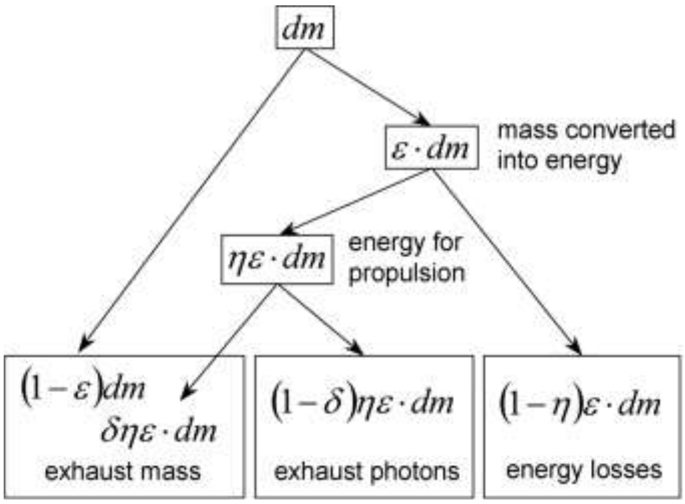


Fig. 6.5: Energy scheme for a relativistic rocket with energy losses and expelled propulsion mass and photons (extracted from [91])

Further information on this topic can be found in the following literature [91,92].