Annex D: Calculation of momentum for relativistic non-elastic collision

During ideal non-elastic, i.e. plastic collision 2 masses hit each other in central position and are moving forward as a combined body without rotation. An approximation procedure is developed to calculate the end-velocity of this body on basis of the principle of conservation of momentum, in a case where the validity of equation $m_3 = m_1 + m_2$ is postulated. This approach is relevant for theoretical analysis only because it can be shown, that in real cases an additional increase of mass Δm_3 because the conversion of potential energy into mass must be considered. For details it is referred to chapter 7.1.

In addition, the appearing simple equation makes it possible to perform a comparison between the approximation procedures recursion, Newton's calculus, and bijection. The latter proved to be superior to the others because it is the only calculation to cover all possible input values and is therefore used also in other calculations in Annex A – C.

Respecting the above-mentioned restrictions, for the relativistic momentum using relation $m_3 = m_1 + m_2$ referring to Eq. (7.01)

$$p_0 = m_1\gamma_1 v_1 + m_2\gamma_2 v_2 = (m_1 + m_2)\gamma_3 v_3 \tag{D.01}$$

applies, where v_3 can be calculated on basis of numerical approximation. In the following different procedures will be presented and the results are compared.

D.1 Recursion procedure

The procedure with the smallest mathematical effort is the procedure using simple recursion. The equation for the development can be derived directly using Eq. (D.01) and shows the form

$$\frac{(v_3)_{k+1}}{c} = \frac{p_0}{c(m_1 + m_2)\gamma_{3k}} = \frac{p_0}{c(m_1 + m_2)}\sqrt{1 - \left(\frac{(v_3)_k}{c}\right)^2} \tag{D.02}$$

D.2 Procedure according to Newton's calculus

Iteration according to Newton' calculus is generally using the sequence

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \tag{D.03}$$

When Eq. (D.01) is converted it applies first

$$\frac{m_1\gamma_1 v_1 + m_2\gamma_2 v_2}{m_1 + m_2} - \gamma_3 v_3 = 0 = f(v_3) \tag{D.04}$$

and then

$$f\left(\frac{v_3}{c}\right) = \frac{p_0}{c(m_1 + m_2)} - \frac{v_3}{c}\left(1 - \frac{{v_3}^2}{c^2}\right)^{-1/2} \quad (D.05)$$

Using

$$x = \frac{v_3}{c} \quad (D.06)$$

it yields

$$f(x) = \frac{p_0}{(m_1 + m_2)} - x(1 - x^2)^{-1/2} \quad (D.07)$$

and

$$f'(x) = -(1 - x^2)^{-3/2} \quad (D.08)$$

After inserting the result in Eq. (D.03) the iteration formula is finally

$$\frac{(v_3)_{k+1}}{c} = \frac{(v_3)_k}{c} + \frac{\frac{p_0}{(m_1 + m_2)} - (v_3)_k\left[1 - \left(\frac{(v_3)_k}{c}\right)^2\right]^{-1/2}}{c\left[1 - \left(\frac{(v_3)_k}{c}\right)^2\right]^{-3/2}} \quad (D.09)$$

D.3 Bisection method

First the starting function is defined using Eq. (D.01)

$$f(v_3) = \gamma_3 v_3 = \frac{p_0}{(m_1 + m_2)} \quad (D.10)$$

where the value for p_0 is defined by the initial starting conditions. For the beginning of the calculation appropriate values for $(v_{3+})_0$ and $(v_{3-})_0$ are determined which are following the conditions

$$f(v_{3+})_0 > \frac{p_0}{(m_1 + m_2)} \quad (D.11)$$

and

$$f(v_{3-})_0 < \frac{p_0}{(m_1 + m_2)} \quad (D.12)$$

In the interval $[(v_{3-})_0; (v_{3+})_0]$ the function $f(v_3)$ must be continuous and differentiable, and further $f'(v_3) \neq 0$ is required, which means, that in the chosen interval no minima and maxima are allowed, because otherwise no exact solution exists. Then the mean value is formed

$$(v_3)_1 = \frac{(v_{3+})_0 + (v_{3-})_0}{2} \quad (D.13)$$

and $f(v_3)_1$ is calculated according to (D.10). The following equations apply:

$$f(v_3)_1 > \frac{p_0}{(m_1 + m_2)} :\Rightarrow (v_{3+})_1 = (v_3)_1 \; und \; (v_{3-})_1 = (v_{3-})_0 \quad (D.14)$$

$$f(v_3)_1 < \frac{p_0}{(m_1 + m_2)} :\Rightarrow (v_{3+})_1 = (v_{3+})_0 \; und \; (v_{3-})_1 = (v_3)_1 \quad (D.15)$$

The calculation is repeated with increasing index 1 to n until the required accuracy is achieved. Every step of the calculation is generating a bisection of the difference between v_{3+} and v_{3-}. A standard calculation program (e.g. Microsoft Excel©) with the utilization of 15 digits is therefore requiring, because of the general estimation

$$2^{10} = 1024 \approx 10^3 \tag{D.16}$$

following

$$10^{15} \approx 2^{50} \tag{D.17}$$

the use of approximately 50 steps to reach maximum possible accuracy; in practice a utilization of 60 proved to be safe in any case. Because of the boundary condition $v_1 = 0$ the starting values can be determined easily and are $(v_{3-})_0 = 0$ resp. $(v_{3+})_0 = v_2$.

D.4 Evaluation

In the following results for the discussed procedures are presented using different values for the velocities. According to the considerations in chapter 7.1 only cases will be viewed, where the masses are equal and one of the selected velocities (here v_1) is equal to zero. All iteration methods lead to the same values; the procedures using simple recursion and according to Newton share the advantage, that they converge very quickly for small values of v/c. However, as a drawback the convergence is reducing for increasing v/c and starting with $p_0 = \gamma_2 v_2 \geq 2c$ (for $m_1 = m_2 = 1$) calculations are no longer possible. Bisection, however, shows a much better performance and is above approx. $v_2/c > 0{,}895$ the only remaining procedure which is still working.

In the following table examples for calculations with different conditions are presented. In all cases it is marked, from which iteration step on no differences between consecutive steps can be detected and so the procedure has reached its end (Status "x" in field "St"). If one of the procedures is not converging, then it is marked as "not ok" in the evaluation field. Further on the differences to the results of the relativistic addition of velocities $v_{3,Rel}$ are presented as percentage-value.

For the calculation, the following equations are used:

$$\frac{v_{3,Rel}}{c} = \frac{1 - \sqrt{1 - \left(\frac{v_2}{c}\right)^2}}{\frac{v_2}{c}} \qquad \gamma_{3,Rel} = \frac{1}{\sqrt{1 - \left(\frac{v_{3,Rel}}{c}\right)^2}} \qquad \frac{p_0}{c} = m_2 \gamma_2 \frac{v_2}{c}$$

Recursion: $$\frac{(v_3)_{k+1}}{c} = \frac{p_0}{c(m_1 + m_2)} \sqrt{1 - \left(\frac{(v_3)_k}{c}\right)^2}$$

Newton

$$\frac{(v_3)_{k+1}}{c} = \frac{(v_3)_k}{c} + \left\{ \frac{p_0}{c(m_1 + m_2)} - \frac{(v_3)_k}{c} \left[1 - \left(\frac{(v_3)_k}{c}\right)^2 \right]^{-1/2} \right\} \left[1 - \left(\frac{(v_3)_k}{c}\right)^2 \right]^{3/2}$$

Bisection: $\frac{(v_3)_{k+1}}{c} = \frac{(v_{3-})_k + (v_{3+})_k}{2c}$

Condition $f(v_3)_{k+1} > \frac{p_0}{(m_1 + m_2)} :\Rightarrow (v_{3+})_{k+1} = (v_3)_{k+1}\ and\ (v_{3-})_{k+1} = (v_{3-})_k$

Condition $f(v_3)_{k+1} < \frac{p_0}{(m_1 + m_2)} :\Rightarrow (v_{3+})_{k+1} = (v_{3+})_k\ and\ (v_{3-})_{k+1} = (v_3)_{k+1}$

Appropriate starting values: For $\frac{(v_{3-})_0}{c} = -\frac{v_1}{c}$ and for $\frac{(v_{3+})_0}{c} = \frac{v_1}{c}$

Values in the fields for results (blue color): For Recursion, Newton and Bisection the last values of iteration.

$\frac{v_3}{v_{3,Rel}} - 1$ Comparison of results. Chosen was bisection (v_3) and relativistic addition of velocities ($v_{3,Rel}$)

For the presented calculations, the following values apply:

Tab. D.1	Tab D.2	Tab D.3
$m_1 = 1;\ m_2 = 1$ $v_1 = 0\ ;\ v_2 = 0{,}1c$	$m_1 = 1;\ m_2 = 1$ $v_1 = 0\ ;\ v_2 = 0{,}8c$	$m_1 = 1:\ m_2 = 1$ $v_1 = 0\ ;\ v_2 = 0{,}89c$

Codes for calculation:

Coordinate		Code
G1	=	(1-SQRT(1-B2*B2))/B2
G2	=	1/SQRT(1-G1*G1)
B3	=	B2/SQRT(1-B2*B2)
B5	=	IF(B6="ok";B70;"")
D5	=	IF(D6="ok";D70;"")
F5	=	IF(F6="ok";F70;"")
H5	=	F5/G1-1
B6	=	IF(C70="";"not ok";"ok")
D6	=	IF(E70="";"not ok";"ok")
F6	=	IF(G70="";"not ok";"ok")
B8	=	B70/D70-1
D8	=	D70/F70-1
F8	=	F70/B70-1
G10	=	B1
H10	=	B2
B11	=	B$3/(1+D$2)*SQRT(1-B10*B10)
C11	=	IF(B11=B10);"x";"")
D11	=	D10+(B$3/(1+D$2)-D10*(1-D10*D10)^-(1/2))*((1-D10*D10)^(3/2))
E11	=	IF(D11=D10);"x";"")
F11	=	(G10+H10)/2
G11	=	IF(F11/SQRT(1-F11*F11)<B$3/(1+D$2);F11;G10)
H11	=	IF(F11/SQRT(1-F11*F11)<B$3/(1+D$2);H10;F11)
I11	=	IF(F11=F10;"x";"")

The codes B11 to I11 to be copied as far as B70 to I70

	A	B	C	D	E	F	G	H	I
1	v_1/c =	0		m_2/m_1=		$v_{3,Rel}$/c =	0,05012563		
2	v_2/c =	0,1		1		$\gamma_{3,Rel}$ =	1,00125866		
3	p_0/c =	0,1005037815							
4		Recursion		Newton		Bisection			
5	v_3/c =	0,0501885613		0,0501885613		0,0501885613	$v_3/v_{3,Rel}$ =	0,1%	
6		nicht ok		ok		ok			
7		Recursion/Newton		Newton/Bisection		Bisection/Recursion			
8		0,0E+00		7,1E-15		-7,2E-15			
9	k	v_3/c	St	v_3/c	St	$(v_{3-}+v_{3+})/2c$	v_{3-}/c	v_{3+}/c	St
10	0	0		0			0	0,1	
11	1	0,0502518908		0,0502518908		0,0500000000	0,0500000000	0,1000000000	
12	2	0,0501884013		0,0501885616		0,0750000000	0,0500000000	0,0750000000	
13	3	0,0501885617		0,0501885613		0,0625000000	0,0500000000	0,0625000000	
14	4	0,0501885613		0,0501885613	x	0,0562500000	0,0500000000	0,0562500000	
15	5	0,0501885613		0,0501885613	x	0,0531250000	0,0500000000	0,0531250000	
16	6	0,0501885613		0,0501885613	x	0,0515625000	0,0500000000	0,0515625000	
17	7	0,0501885613	x	0,0501885613	x	0,0507812500	0,0500000000	0,0507812500	
18	8	0,0501885613	x	0,0501885613	x	0,0503906250	0,0500000000	0,0503906250	
19	9	0,0501885613	x	0,0501885613	x	0,0501953125	0,0500000000	0,0501953125	
20	10	0,0501885613	x	0,0501885613	x	0,0500976563	0,0500976563	0,0501953125	
21	11	0,0501885613	x	0,0501885613	x	0,0501464844	0,0501464844	0,0501953125	
22	12	0,0501885613	x	0,0501885613	x	0,0501708984	0,0501708984	0,0501953125	
23	13	0,0501885613	x	0,0501885613	x	0,0501831055	0,0501831055	0,0501953125	
24	14	0,0501885613	x	0,0501885613	x	0,0501892090	0,0501831055	0,0501892090	
25	15	0,0501885613	x	0,0501885613	x	0,0501861572	0,0501861572	0,0501892090	
26	16	0,0501885613	x	0,0501885613	x	0,0501876831	0,0501876831	0,0501892090	
27	17	0,0501885613	x	0,0501885613	x	0,0501884460	0,0501884460	0,0501892090	
28	18	0,0501885613	x	0,0501885613	x	0,0501888275	0,0501884460	0,0501888275	
29	19	0,0501885613	x	0,0501885613	x	0,0501886368	0,0501884460	0,0501886368	
30	20	0,0501885613	x	0,0501885613	x	0,0501885414	0,0501885414	0,0501886368	
31	21	0,0501885613	x	0,0501885613	x	0,0501885891	0,0501885414	0,0501885891	
32	22	0,0501885613	x	0,0501885613	x	0,0501885653	0,0501885414	0,0501885653	
33	23	0,0501885613	x	0,0501885613	x	0,0501885533	0,0501885533	0,0501885653	
34	24	0,0501885613	x	0,0501885613	x	0,0501885593	0,0501885593	0,0501885653	
35	25	0,0501885613	x	0,0501885613	x	0,0501885623	0,0501885593	0,0501885623	
36	26	0,0501885613	x	0,0501885613	x	0,0501885608	0,0501885608	0,0501885623	
37	27	0,0501885613	x	0,0501885613	x	0,0501885615	0,0501885608	0,0501885615	
38	28	0,0501885613	x	0,0501885613	x	0,0501885612	0,0501885612	0,0501885615	
39	29	0,0501885613	x	0,0501885613	x	0,0501885613	0,0501885612	0,0501885613	
40	30	0,0501885613	x	0,0501885613	x	0,0501885612	0,0501885612	0,0501885613	
41	31	0,0501885613	x	0,0501885613	x	0,0501885613	0,0501885613	0,0501885613	
42	32	0,0501885613	x	0,0501885613	x	0,0501885613	0,0501885613	0,0501885613	
60	50	0,0501885613	x	0,0501885613	x	0,0501885613	0,0501885613	0,0501885613	
61	51	0,0501885613	x	0,0501885613	x	0,0501885613	0,0501885613	0,0501885613	
62	52	0,0501885613	x	0,0501885613	x	0,0501885613	0,0501885613	0,0501885613	x
63	53	0,0501885613	x	0,0501885613	x	0,0501885613	0,0501885613	0,0501885613	x
64	54	0,0501885613	x	0,0501885613	x	0,0501885613	0,0501885613	0,0501885613	x
65	55	0,0501885613	x	0,0501885613	x	0,0501885613	0,0501885613	0,0501885613	x

Tab. D.1: Velocity v_3 after relativistic non-elastic collision, $v_1 = 0$; $v_2 = 0{,}1c$

	A	B	C	D	E	F	G	H	I
1	v_1/c =	0		m_2/m_1=		$v_{3,Rel}$/c =	0,50000000		
2	v_2/c =	0,8		1		$\gamma_{3,Rel}$ =	1,15470054		
3	p_0/c =	1,3333333333							
4		Recursion		Newton		Bisection			
5	v_3/c =	0,5547001962		0,5547001962		0,5547001962	$v_3/v_{3,Rel}$ =	10,9%	
6		ok		ok		ok			
7		Recursion/Newton		Newton/Bisection		Bisection/Recursion			
8		0,0E+00		4,7E-15		-4,6E-15			
9	k	v_3/c	St	v_3/c	St	$(v_{3-}+v_{3+})/2c$	v_{3-}/c	v_{3+}/c	St
10	0	0		0			0	0,8	
11	1	0,6666666667		0,6666666667		0,4000000000	0,4000000000	0,8000000000	
12	2	0,4969039950		0,5723540713		0,6000000000	0,4000000000	0,6000000000	
13	3	0,5785370130		0,5550845393		0,5000000000	0,5000000000	0,6000000000	
14	4	0,5437707542		0,5547003739		0,5500000000	0,5500000000	0,6000000000	
15	5	0,5594891983		0,5547001962		0,5750000000	0,5500000000	0,5750000000	
16	6	0,5525584281		0,5547001962		0,5625000000	0,5500000000	0,5625000000	
17	7	0,5556494433		0,5547001962	x	0,5562500000	0,5500000000	0,5562500000	
18	8	0,5542777868		0,5547001962	x	0,5531250000	0,5531250000	0,5562500000	
19	9	0,5548878305		0,5547001962	x	0,5546875000	0,5546875000	0,5562500000	
20	10	0,5546167828		0,5547001962	x	0,5554687500	0,5546875000	0,5554687500	
21	11	0,5547372648		0,5547001962	x	0,5550781250	0,5546875000	0,5550781250	
22	12	0,5546837205		0,5547001962	x	0,5548828125	0,5546875000	0,5548828125	
23	13	0,5547075186		0,5547001962	x	0,5547851563	0,5546875000	0,5547851563	
24	14	0,5546969418		0,5547001962	x	0,5547363281	0,5546875000	0,5547363281	
25	15	0,5547016426		0,5547001962	x	0,5547119141	0,5546875000	0,5547119141	
26	16	0,5546995534		0,5547001962	x	0,5546997070	0,5546997070	0,5547119141	
27	17	0,5547004819		0,5547001962	x	0,5547058105	0,5546997070	0,5547058105	
28	18	0,5547000692		0,5547001962	x	0,5547027588	0,5546997070	0,5547027588	
29	19	0,5547002527		0,5547001962	x	0,5547012329	0,5546997070	0,5547012329	
30	20	0,5547001711		0,5547001962	x	0,5547004700	0,5546997070	0,5547004700	
31	21	0,5547002074		0,5547001962	x	0,5547000885	0,5547000885	0,5547004700	
32	22	0,5547001913		0,5547001962	x	0,5547002792	0,5547000885	0,5547002792	
33	23	0,5547001984		0,5547001962	x	0,5547001839	0,5547001839	0,5547002792	
34	24	0,5547001952		0,5547001962	x	0,5547002316	0,5547001839	0,5547002316	
35	25	0,5547001967		0,5547001962	x	0,5547002077	0,5547001839	0,5547002077	
36	26	0,5547001960		0,5547001962	x	0,5547001958	0,5547001958	0,5547002077	
37	27	0,5547001963		0,5547001962	x	0,5547002017	0,5547001958	0,5547002017	
38	28	0,5547001962		0,5547001962	x	0,5547001988	0,5547001958	0,5547001988	
39	29	0,5547001962		0,5547001962	x	0,5547001973	0,5547001958	0,5547001973	
40	30	0,5547001962		0,5547001962	x	0,5547001965	0,5547001958	0,5547001965	
41	31	0,5547001962		0,5547001962	x	0,5547001962	0,5547001962	0,5547001965	
42	32	0,5547001962		0,5547001962	x	0,5547001963	0,5547001962	0,5547001963	
60	50	0,5547001962	x	0,5547001962	x	0,5547001962	0,5547001962	0,5547001962	
61	51	0,5547001962	x	0,5547001962	x	0,5547001962	0,5547001962	0,5547001962	x
62	52	0,5547001962	x	0,5547001962	x	0,5547001962	0,5547001962	0,5547001962	x
63	53	0,5547001962	x	0,5547001962	x	0,5547001962	0,5547001962	0,5547001962	
64	54	0,5547001962	x	0,5547001962	x	0,5547001962	0,5547001962	0,5547001962	x
65	55	0,5547001962	x	0,5547001962	x	0,5547001962	0,5547001962	0,5547001962	x

Tab. D.2: Velocity v_3 after relativistic non-elastic collision, $v_1 = 0$; $v_2 = 0{,}8c$

	A	B	C	D	E	F	G	H	I
1	v_1/c =	0		m_2/m_1=		$v_{3,Rel}$/c =	0,61128031		
2	v_2/c =	0,89		1		$\gamma_{3,Rel}$ =	1,26356090		
3	p_0/c =	1,9519233617							
4		Recursion		Newton		Bisection			
5	v_3/c =			0,6984528781		0,6984528781	$v_3/v_{3,Rel}$ =	14,3%	
6		not ok		ok		ok			
7		Recursion/Newton		Newton/Bisection		Bisection/Recursion			
8		3,4E-02		0,0E+00		-3,3E-02			
9	k	v_3/c	St	v_3/c	St	$(v_{3-}+v_{3+})/2c$	v_{3-}/c	v_{3+}/c	St
10	0	0		0			0	0,89	
11	1	0,9759616809		0,9759616809		0,4450000000	0,4450000000	0,8900000000	
12	2	0,2127032246		0,9397078220		0,6675000000	0,6675000000	0,8900000000	
13	3	0,9536286032		0,8688424449		0,7787500000	0,6675000000	0,7787500000	
14	4	0,2937506647		0,7743135001		0,7231250000	0,6675000000	0,7231250000	
15	5	0,9329042795		0,7115556340		0,6953125000	0,6953125000	0,7231250000	
16	6	0,3514676442		0,6988106129		0,7092187500	0,6953125000	0,7092187500	
17	7	0,9136953543		0,6984531401		0,7022656250	0,6953125000	0,7022656250	
18	8	0,3966306344		0,6984528781		0,6987890625	0,6953125000	0,6987890625	
19	9	0,8959116343		0,6984528781		0,6970507813	0,6970507813	0,6987890625	
20	10	0,4335537100		0,6984528781	x	0,6979199219	0,6979199219	0,6987890625	
21	11	0,8794661312		0,6984528781	x	0,6983544922	0,6983544922	0,6987890625	
22	12	0,4645201595		0,6984528781	x	0,6985717773	0,6983544922	0,6985717773	
23	13	0,8642751101		0,6984528781	x	0,6984631348	0,6983544922	0,6984631348	
24	14	0,4909276759		0,6984528781	x	0,6984088135	0,6984088135	0,6984631348	
25	15	0,8502581396		0,6984528781	x	0,6984359741	0,6984359741	0,6984631348	
26	16	0,5137129813		0,6984528781	x	0,6984495544	0,6984495544	0,6984631348	
27	17	0,8373381377		0,6984528781	x	0,6984563446	0,6984495544	0,6984563446	
28	18	0,5335439275		0,6984528781	x	0,6984529495	0,6984495544	0,6984529495	
29	19	0,8254414098		0,6984528781	x	0,6984512520	0,6984512520	0,6984529495	
30	20	0,5509184645		0,6984528781	x	0,6984521008	0,6984521008	0,6984529495	
31	21	0,8144976752		0,6984528781	x	0,6984525251	0,6984525251	0,6984529495	
32	22	0,5662205832		0,6984528781	x	0,6984527373	0,6984527373	0,6984529495	
33	23	0,8044400793		0,6984528781	x	0,6984528434	0,6984528434	0,6984529495	
34	24	0,5797542286		0,6984528781	x	0,6984528965	0,6984528434	0,6984528965	
35	25	0,7952051896		0,6984528781	x	0,6984528700	0,6984528700	0,6984528965	
36	26	0,5917650167		0,6984528781	x	0,6984528832	0,6984528700	0,6984528832	
37	27	0,7867329748		0,6984528781	x	0,6984528766	0,6984528766	0,6984528832	
38	28	0,6024547712		0,6984528781	x	0,6984528799	0,6984528766	0,6984528799	
39	29	0,7789667662		0,6984528781	x	0,6984528782	0,6984528766	0,6984528782	
40	30	0,6119916160		0,6984528781	x	0,6984528774	0,6984528774	0,6984528782	
41	31	0,7718532028		0,6984528781	x	0,6984528778	0,6984528778	0,6984528782	
42	32	0,6205171991		0,6984528781	x	0,6984528780	0,6984528780	0,6984528782	
60	50	0,6671025921		0,6984528781	x	0,6984528781	0,6984528781	0,6984528781	
61	51	0,7270581323		0,6984528781	x	0,6984528781	0,6984528781	0,6984528781	x
62	52	0,6700717735		0,6984528781	x	0,6984528781	0,6984528781	0,6984528781	x
63	53	0,7244527587		0,6984528781	x	0,6984528781	0,6984528781	0,6984528781	x
64	54	0,6727542511		0,6984528781	x	0,6984528781	0,6984528781	0,6984528781	x
65	55	0,7220808780		0,6984528781	x	0,6984528781	0,6984528781	0,6984528781	x

Tab. D.3: Velocity v_3 after relativistic non-elastic collision, $v_1 = 0$; $v_2 = 0{,}89c$