Annex E: Brief introduction to vector calculus

To understand the representation of Maxwell's equations in Chapter 10, a basic knowledge of vector calculus is required. The necessary relationships and basic elements for understanding field relationships are summarized here in brief. Only the absolutely necessary relationships are shown, and the following restrictions apply:

1. The representations apply to 3 dimensions; these are sufficient for the relationships in fields.
2. Only Cartesian (rectangular) coordinate systems are considered (e.g. no spherical or cylindrical coordinates).

First, the basic properties of vectors are presented and then the differential functions required to understand Maxwell's equations are explained.

E.1 Scalar und Vector

In a coordinate system, physical quantities can be assigned to each point as a scalar or vector. Vectors are direction-dependent, scalars are not. Examples of scalar quantities are temperature, energy, and pressure. For directional quantities such as forces or fields, on the other hand, vectors are used which, in addition to the location in the coordinate system, also contain values for the magnitude and direction. For the representation of a vector $\vec{a}$ in Cartesian coordinates the following form is used:

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \qquad \text{(E.01)}$$

The amount of $\vec{a}$, for example for the magnitude of a force, is determined by

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad \text{(E.02)}$$

If the direction and magnitude of two vectors are the same, they are identical, but can be located at different points in the coordinate system.

E.2 Vector addition

For the addition of two vectors $\vec{a}$ and $\vec{b}$ the rule applies:

$$\vec{a} + \vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{pmatrix} \qquad (E.03)$$

This addition can also be performed graphically. For this purpose, a representation with arrows is used. The position in the diagram is the direction, the length of the arrow indicates the magnitude.

For the addition, the arrows $\vec{a}$ and $\vec{b}$ are joined together; the resulting line between the start and end points is the result of the addition in terms of magnitude and direction.

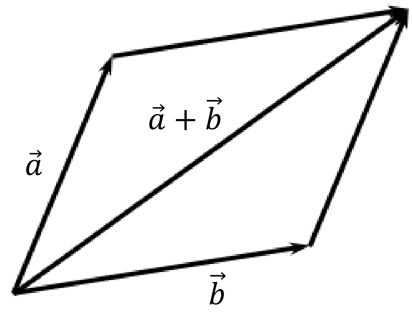

Fig.. E.1: Graphical vector addition

E.3 Scalar product

The scalar product (or inner product) of two vectors is so called because the result of the multiplication is a scalar. This is in Cartesian coordinates

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z \qquad (E.04)$$

or

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi \qquad (E.05)$$

with φ as the angle between $\vec{a}$ and $\vec{b}$. This operation is often used in physics when energy is to be calculated and the angle between the force and the direction of movement does not match. Force and direction are vectors, the resulting work is a scalar quantity. The meaning becomes clear when a mass in the Earth's gravitational field and an attacking force is considered. If the mass is moved upwards by the force ($\varphi = 0; \cos\varphi = 1$), energy is needed and the potential energy increases; if the force acts at $\varphi = 90°$, the mass remains at the same height and the energy does not change.

E.4 Cross product

The cross product (also known as the vector product or outer product) of the vectors $\vec{a}$ and $\vec{b}$ in three-dimensional space is a certain vector that is perpendicular to the plane spanned by them. The length is equal to the area of the parallelogram, i.e.

$$\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot |\sin\varphi| \qquad (E.06)$$

In the three-dimensional Cartesian coordinate system, the cross product is calculated as follows

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix} \tag{E.07}$$

Examples of the application of the cross product are the Lorentz force or the torque. For example, the following relationship applies to the magnetic part of the Lorentz force

$$\vec{F}_B = q\vec{v} \times \vec{B} \tag{E.08}$$

with q as the charge and $\vec{v}$ as its velocity and $\vec{B}$ as the magnetic field. The orientation of the resulting Lorentz force is perpendicular to both the velocity and the magnetic field (3-finger rule).

E.5 Fields and Nabla operator

In physics, a field is defined as the spatial distribution of a physical quantity. In the simplest case, there is a scalar field, as is possible for temperature distributions or potentials. If a physical vector is dependent on the position of the location, it is referred to as a vector field. It can be visualized by field lines, whereby the tangent to the field line indicates the direction of the vector. The magnitude of the vector is represented by the density of the field lines. Electric and magnetic fields are examples of this. These fields are characterized by the fact that temporal changes in particular play a role, which must be represented by differentiation. The use of the Nabla operator is helpful here.

The Nabla operator $\vec{\nabla}$ is a vectorial differential operator. This means that it can be written in vector form and, when applied to a function, performs a differential operation that represents a 3-dimensional derivative. With its help, the quantities gradient, divergence, and rotation, which are still to be described, can be easily represented. It is defined for the 3-dimensional Cartesian coordinates x, y, z as

$$\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{E.09}$$

E.6 Gradient

A field based on a scalar function f assigns an exact value to each point in the definition space. Examples of scalar fields in three-dimensional space are the distribution of temperatures, density, or potentials. Applying the Nabla operator to f results in a vector field called the gradient (grad). The gradient points in the direction of the strongest ascent at each point in space and its magnitude indicates the increase in this direction. The representation is as follows:

$$\text{grad } f = \vec{\nabla} \cdot f \;\; = \begin{pmatrix} \frac{\partial f_x}{\partial x} \\ \frac{\partial f_y}{\partial y} \\ \frac{\partial f_z}{\partial z} \end{pmatrix} \tag{E.10}$$

If the scalar field is a potential, the negative gradient of the field indicates the associated force field. This is clear in the case of the gravitational field: Two of the coordinates are equal to zero and a body falls in the direction in which the change in its potential reaches the maximum.

E.7 Divergence

When applying the Nabla operator to a vector field f, the scalar product $\vec{\nabla} \cdot f$ results in a scalar field that indicates whether field lines appear or disappear at each point in space. Thus, at the location of a positive charge, the divergence of the electric field is greater than zero, as field lines arise at this point. Points with positive divergence are called sources, points with negative divergence are called sinks. The calculation results in

$$\text{div } \vec{f} = \vec{\nabla} \cdot \vec{f} = \; = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \tag{E.11}$$

E.8 Rotation

If we form $\vec{\nabla} \times f$, we obtain a vector function called rotation (rot), which characterizes the closed loop of the vector field f. If we consider, for example, the magnetic field of a current-carrying wire, the field lines run in a circle around this wire and are closed. The calculation is carried out as follows:

$$\text{rot } \vec{f} = \vec{\nabla} \times \vec{f} = \begin{pmatrix} \frac{\partial}{\partial y} f_z - \frac{\partial}{\partial z} f_y \\ \frac{\partial}{\partial z} f_x - \frac{\partial}{\partial x} f_z \\ \frac{\partial}{\partial x} f_y - \frac{\partial}{\partial y} f_x \end{pmatrix} \tag{E.12}$$